3. General Theoretical Foundation

Conflict Graph $G(V, E)$

- V: configurations in $H(N^i)$ and objects in $N \setminus N^i$
- E: conflict relations between configurations in $H(N^i)$ and objects in $N \setminus N^i$

1. Use the conflict graph to find out all configurations in $H(N^i)$ which conflict with S^{i+1}
2. Create new configrautions defined (or called supported) by S^{i+1}
3. Update conflict graph
   - Remove invalid configurations and the corresponding edges
   - Add edges between the new configuations in $H(N^{i+1})$ and their conflicted objectes in $N \setminus N^{i+1}$

Hisotry graph $G(V, E)$ (directed graph)

- V: configurations in $H(N^0)$, $H(N^1)$, ..., $H(N^i)$
- E: direct arcs from $H(N^{j-1}) \setminus H(N^j)$ and $H(N^j) \setminus H(N^{j-1})$, for $1 \leq j \leq i$, i.e., configurations killed by S^j and configuartions created by S^j
  - G is an acyclic graph, and only configuartions in $H(N^0)$ don't have in-going edges and are called roots.
  - If an object S conflicts with a configuration f, there is one path from a root to f along which all configuartions are in conflict with S.
  - (optional) Each configuration has a constant number of out-going edges.

1. Use the history to find out all configurations in $H(N^i)$ in conflict with S^{i+1}
2. Create new configrautions defined (or called supported) by S^{i+1}
3. Add edges between $H(N^i) \setminus H(N^{i+1})$ and $H(N^{i+1}) \setminus H(N^i)$

Kenneth L. Clarkson, Kurt Mehlhorn, and Raimund Seidel

Four, Results on Randomized Incremental Construction

Cpmputational Geometry: Theory and Applications 3, pp. 185–212, 1993.

Denotation Changes

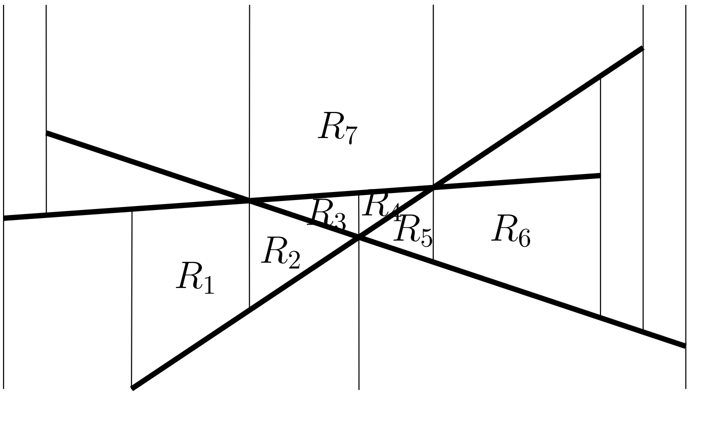
N	S
$S_1, S_2, \ldots, S_n$	$\pi_S = x_1, x_2, \ldots, x_n$
$H(N)$	$\mathcal{F}_0(S)$
$history(i)$	$H(i)$

3.1 Basic Denotations

$\boldsymbol{S}$: a set of n objects (points, line segments, circles)

$\mathcal{F}(S)$: configurations defined by S

- A configuration is defined by at most b objects.
  - a triangle is defined by 3 points, a trapezoid is defined by at most 4 line segments.
- A multiset: $c \leq b$ elements can define more than one configuration
  - 3 segments can defined 7 trapezoids
- For a configuration $F \in \mathcal{F}(S)$ and an object $x \in S$, if $x \in F$, F replies on x and x supports F



$C \subseteq S \times \mathcal{F}(S)$: conflict relations between S and $\mathcal{F}(S)$

- $(x, F) \in C \to x$ does not support F
- $(x, F) \in C$ usually means a nonempty intersection between x and F
  - a point x insides a triangle F

Example: Vertical Trapezoidal decomposition

- S: a set of n line segment
- $\mathcal{F}(S)$: trapezoids defined by S (two trapezoids can intersect)
- $(x, F) \in C$: line segment x intersects F
  - Different from that an endpoint of x is located inside a trapezoid F

$\boldsymbol{\mathcal{F}_0(R)} = \{F \in \mathcal{F}(R) \mid \forall x \in R, (x, F) \notin C\}$, for a r-element random sample R of S

- any configuration in $\mathcal{F}_0(R)$ does not conflict with any object in R.

$\pi = (x_1, x_2, \ldots, x_n)$ is a random permutation of S

- $R_j = \{x_1, x_2, \ldots, x_j\}$
- $\pi_r = (x_1, x_2, \ldots, x_j)$

History $H_r(\pi) = H(x_1, x_2, \ldots, x_r) = \bigcup_{1 \leq i \leq r} \mathcal{F}_0(R_i)$

- $(x_1, x_2, \ldots, x_r)$ is the first r elements of π_S
- equivalent to trapzoids in history(r)
- $H_r = H_r(\pi)$

Fact

If $\pi = (x_1, x_2, \ldots, x_n)$ is a random permutation of S, R_j is a random subset of size j of S, $(x_1, x_2, \ldots, x_j)$ is a random permutation of R_j, x_j is a random element of R_j, and if δ is a (fixed) permutation, $\pi\delta$ is random permutation

For a subset $R \subseteq S$, $r = |R|$, and two distinct objects, $x, y, \in R$,

- $\deg(x, R) = |\{F \in \mathcal{F}_0(R) \mid x \text{ supports } F\}|$
  - the number of triangles in a triangulation incident to a point x
- $\mathrm{pdeg}(x, y, R) = |\{F \in \mathcal{F}_0(R) \mid x \text{ and } y \text{ support } F\}|$
  - the number of triangles in a triangulation iwith an edge $\overline{xy}$
- $c(R) = \frac{1}{r} \sum_{x \in R} \deg(x, R)$
- $p(R) = \frac{1}{r(r-1)} \sum_{(x,y) \in R \times R} \mathrm{pdeg}(x, y, R)$

Important Expected Values

- $c_r = E[c(R)] = \sum_{R \subseteq S, |R|=r} c(R) / \binom{n}{r}$
- $p_r = E[p(R)] = \sum_{R \subseteq S, |R|=r} p(R) / \binom{n}{r}$
- $f_r = \sum_{R \subseteq S, |R|=r} |\mathcal{F}_0(R)| / \binom{n}{r}$
- $c_1 = p_1 = f_1$ and for $j < 1$ or $j > n$, $c_j = p_j = f_j = 0$.

3.2 Lemmas and Theorems

All expected values are computed with respect to a random permutation $\pi = (x_1, x_2, \ldots, x_n)$ of S

Lemma 1

1. $c_r \leq b f_r / r$
2. $p_r \leq b(b-1) f_r / r(r-1)$, for $r > 1$

proof: For evey configuration $F \in \mathcal{F}_0(S)$

1. At most b objects support F
2. At most $b(b-1)$ order pairs of objects support F

Theorem 1
Let C_r be the expected size of H_r. $C_r = \sum_{1\le i\le r} c_i$.
proof:

1. H_0 is empty and $C_0 = 0$
2. For $i \ge 1$, $|H_i \setminus H_{i-1}| = \deg(x_i, R_i)$.
3. R_i is a randome subset of S of size i and x_i is a random element of R_i, $E[\deg(x_i, R_i)] = E[c(R_i)] = c_i$.
4. $E[|H_r|] = E[\sum_{1\le i\le n} |H_i \setminus H_{i-1}|] = \sum_{1\le i\le n} |E[|H_i \setminus H_{i-1}|] = \sum_{1\le i\le r} c_i$

Example Let R be a random subset of S of size r

- Since the triangualtion of R has $O(r)$ triangles, $c_r = O(1)$ and $E[|H_r|] = O(r)$.
- Since the expected number of trapezoids in the trapezoidal decomposition of R is $O(r+kr^2/n^2)$, where k is the number of intersections among the n line segments, $c_r = O(1 + kr/n^2)$ and $E[|H_r|] = O(r + kr^2/n^2)$

Theorem 2
The expected number of configurations in H_{r-1} which are in conflict with x is $-c_r + \sum_{j\le r} p_j$.
proof:

- Let X be the number of configurations $F \in H_{r-1}$ with $(x_r, F) \in C$
- Let $H = H_{r-1} = H(x_1, x_2, \ldots, x_{r-})$
  Let $H' = H(x_r, x_1, \ldots, x_{r-1})$, i.e., x_r is pretend to be inserted first.
- $|H \cup H'| = |H| + |H' \setminus H| = |H'| + |H \setminus H'|$
- $X = |H \setminus H'|$
- $H' \setminus H$ comprises configurations supported by x_r.
  How many of them appear when x_j is inserted, $1 \le j \le r - 1$.
  Let $R'_j = R_j \cup \{x_j\}$. For each $F \in H' \setminus H$,
  - either $F \in \mathcal{F}_0(\{x_r\})$ or
  - $F \in \mathcal{F}_0(R'_j)$ and x_j support F, $\exists j \ge 1$. Since F must be supported by x_r, the total number is $\mathrm{pdeg}(x_r, x_j, R'_j)$

- $X = |H| - |H'| + |H' \setminus H|$
  $= |H| - |H'| + |\mathcal{F}_0(\{x_r\}) + \sum_{1 \le j \le r-1} \mathrm{pdeg}(x_r, x_j, R'_j)$
  $E[X] = E[|H|] - E[|H'|] + E[|\mathcal{F}_0(\{x_r\})| + \sum_{1 \le j \le r-1} E[\mathrm{pdeg}(x_r, x_j, R'_j)]$
- $E[|H|] = C_{r-1}$, $E[|H'|] = C_r$, and $C_{r-1} - C_r = -c_r$
- $E[|\mathcal{F}_0(\{x_r\})|] = f_1 = p_1$ and $E[\mathrm{pdeg}(x_r, x_j, R'_j)] = p_{j+1}$ since R'_j is a random subset of S of size $j+1$ and x_r and x_j are random elements of this subset
- $E[X] = -c_r + \sum_{j \le r} p_j$

Example: Vertical Trapezoidal Decomposition

- $c_i \le bf_i/i = 4 * O(i + ki^2/n^2)/i = O(1 + ki/n^2)$
- $p_i \le b(b-1)f_i/i(i-1) = 12O(i + ki^2/n^2)/i(i-1) = O(1/i + k/n^2)$
- $-O(1 + ki/n^2) + \sum_{1 \le i \le r} O(1/i + k/n^2) = \log i + kr^2/n^2$

Lemma 2

1. The expected number of configurations in $\mathcal{F}_0(R_{j-1})$ in conflict with x_r is $f_{j-1} - f_j + c_j$
2. The expected number of configurations in $\mathcal{F}_0(R_{j-1})$ supported by x_{j-1} and in conflict with x_r is at most $b(f_{j-1} - f_j + c_j)/(j-1)$

proof

1. Difference between $\mathcal{F}_0(R)$ and $\mathcal{F}_0(R \cup \{x\})$
   - configurations in $\mathcal{F}_0(R)$ in conflict with x
   - configuration in $\mathcal{F}_0(R \cup \{x\})$ supported by x_r

   $\mathcal{F}_0(R_{j-1} \cup \{x_r\}) = \mathcal{F}_0(R_{j-1}) \setminus \{F \in \mathcal{F}_0(R_{j-1}) \mid (x_r, F) \in C\} \cup \{F \in \mathcal{F}_0(R_{j-1} \cup \{x_r\}) \mid x_r \text{ supports } F\}$
   $\rightarrow E[|\mathcal{F}_0(R_{j-1})|] - E[|\mathcal{F}_0(R_{j-1} \cup \{x_r\})|] + E[|\{F \in \mathcal{F}_0(R_{j-1} \cup \{x_r\}) \mid x_r \text{ supports } F\}|] = f_{j-1} - f_j + c_j$
2. Since x_{j-1} is a random element of R_{j-1}, the probability with which a configuration in (1) is supported by x_{j-1} is at most $b/(j-1)$, implying an expected value $b(f_{j-1} - f_j + c_j)/(j-1)$

Conflict History

- $G = G_n = G_\pi = C \cap (S \times H_n)$ for a random sequence π of S, i.e., the conflict relations between S and H_n.
- Bipartite Graph $G(U, V, E)$
  - $U = S$
  - $V = H_n$
  - $E = \{(u, v) \mid u \in U, v \in V, (u, v) \in C\}$
- $|G| = |E|$

Theorem 3
$E[|G|] = -C_n + \sum_{1 \le j \le r}(n - j + 1)p_j$.
proof
$E[|G|] = \sum_{1 \le i \le r}(-c_r + \sum_{1 \le j \le i} p_j)$
$= -C_n + \sum_{1 \le i \le r} \sum_{1 \le j \le i} p_j$
$= -C_n + \sum_{1 \le j \le r}(n - j + 1)p_j$ since p_j occurs $(n - j + 1)$ times

Example Vertical Trapezoidal Decomposition

- $C_n = \sum_{1 \le i \le n} O(i + ki/n^2) = O(n + k)$
- $|G| \le \sum_{1 \le i \le n}(n - i + 1)O(1/i + k/n^2)$
  $\le \sum_{1 \le i \le n} O(n/i + k/n) = O(n \log n + k)$
- note that a conflict relation between a segment x and a trapezoid F indictes that x intersect F (not defined for an endpoint of x)

3.3 Deletion

For $\pi = (x_1, x_2, \ldots, x_n) \in \Pi_S$ and $i \in [1 \cdots n]$,
$\pi \setminus i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Delete x_i from π as x_i has never been inserted.

- Compute $H(\pi \setminus i)$ from $H(\pi)$
- Analyze $G(\pi \setminus i)$ from $G(\pi)$

Theorem 4
$\frac{1}{n!n} \sum_{\pi \in \Pi_S} \sum_{1 \le i \le n} |H(\pi) \oplus H(\pi \setminus i)| \le 2b\frac{C_n}{n} - c_n$.
proof

- $|B \oplus A| = |A| - |B| + 2|B \setminus A|$
  $|H \oplus H(\pi \setminus i)| = |H(\pi \setminus i)| - |H| + 2|H \setminus H(\pi \setminus i)|$
- $H \setminus H(\pi \setminus i)$ comprises configurations in H supported by x_i
  - $E[|H|] = C_n$, and any $F \in H$ is supported by no more than b objects
  - $E[|H \setminus H(\pi \setminus i)|] \le bC_n/n$
- $E[|H(\pi) \oplus H(\pi \setminus i)|] = C_{n-1} - C_n + 2E[|H \setminus H(\pi \setminus i)|] \le -c_n + 2bC_n/n$

Example: Vertical Trapezoidal Decomposition

- $C_n = O(k + n)$, $b = 4$, and $c_i = O(1 + ki^2/n^2)$
- $E[|H \oplus H(\pi \setminus i)|] = O(1 + k/n)$

Theroem 5
$E[|G(\pi \setminus i) \setminus G(\pi)|] = \frac{1}{n!n} \sum_{\pi \in \Pi_S} \sum_{1 \leq i \leq n} |G(\pi \setminus i) \setminus G(\pi)|$
$\leq c_n - (b+1)C_n/n + \sum_{1 \leq j \leq n} bp_j - \sum_{1 \leq j \leq n} (b+1)(j-1)p_j/n.$
proof

- $G = G(\pi)$, $|G(\pi \setminus i) \setminus G| = |G(\pi \setminus i)| - |G| + |G \setminus G(\pi \setminus i)|$
  $\to E[|G(\pi \setminus i) \setminus G|] = E[|G(\pi \setminus i)|] - E[|G|] + E[|G \setminus G(\pi \setminus i)|]$
  $\to E[|G(\pi \setminus i) \setminus G|] = E[|G \setminus G(\pi \setminus i)|] + c_n - \sum_{1 \leq j \leq n} p_j$
- A pair (x, F) is in $G \setminus G(\pi \setminus i)$ if it is G and either $x_i = x$ or $x_i \in F$.
  $\to$ at most $b+1$ choices of x_i
  $\to$ the probablity with $(x, F) \in G \setminus G(\pi \setminus i)$ is $b + 1/n$
- $E[|G \setminus G(\pi \setminus i)|] \leq (b+1)E[|G|]/n$

Example: Vertical Trapezoidal Decomposition

- $E[|G \setminus G(\pi \setminus i)|] = O(\log n + k/n)$

Theroem 6
For a fixed i, let I be the set of conflicts of the form (x_j, F) with $j > i$ and $F \in \mathcal{F}_0(R_{i-1}) \setminus \mathcal{F}_0(R_i)$. Then for random $\pi \in \Pi_S$ and random $i \in [1 \cdots n]$, $E[|I|] = (E[|G|] - E[|H|] + f_n)/n$
proof

- Let I_i denote the set I for $x_i \to E[|I|] = \sum_{1 \leq i \leq n} E[|I_i|]/n$
- Since I_i are disjoint, $E[I] = E[|\bigcup_i I_i|]/n$
- For any conflict $(x_j, F) \in G$,
  - either $F \in \mathcal{F}_0(R_{j-1})$
  - or there is exactly one $i < j$ such that $F \in \mathcal{F}_0(R_{i-1}) \setminus \mathcal{F}_0(R_i)$
    $\to (x_j, F) \in I_i$
- $E[|G|] = E[|\bigcup_{1 \leq i \leq n} I_i|] + E[|\{(x_j, F) \in G \mid F \in \mathcal{F}_0(R_{j-1})\}|]$
- For each conflict (x_j, F) with $F \in \mathcal{F}_0(R_{j-1})$, F appears in $H \setminus \mathcal{F}_0(S)$ exactly once $\to E[|\{(x_j, F) \in G \mid F \in \mathcal{F}_0(R_{j-1})\}|] = E[|H|] - |\mathcal{F}_0(S)|$

Example: Vertical Trapezoidal Decomposition

- $E[|I|] = O(\log n + k/n)$