Discrete and Computational Geometry, WS1415
Exercise Sheet "10":
University of Bonn, Department of Computer Science I

- *Written solutions have to be prepared until* **Tuesday 20th of January 14:00 pm**. *There will be a letterbox in the LBH building.*
- *You may work in groups of at most two participants.*
- *Please contact Hilko Delonge, hilko.delonge@uni-bonn.de, if you want to participate and have not yet signed up for one of the exercise groups.*
- *If you are not yet subscribed to the mailing list, please do so at https://lists.iai.uni-bonn.de/mailman/listinfo.cgi/lc-dcgeom*

**Exercise 22: Voronoi edges of $k^{\text{th}}$-order Voronoi diagrams (6 points)**

Consider a Voronoi edge $e$ between two adjacent Voronoi regions $\text{VR}_k(H_1, S)$ and $\text{VR}_k(H_2, S)$, where $S$ is a set of $n$ point sites in the Euclidean plane. Please prove the following.

1. $|H_1 \setminus H_2| = |H_2 \setminus H_1| = 1$
2. The circle centered at a point $x$ in $e$ and touching $p$ and $q$, where $H_1 \setminus H_2 = \{p\}$ and $H_2 \setminus H_1 = \{q\}$, encloses exactly $k-1$ sites of $S$.

(Hint: Consider $\text{VR}_{k-1}(H, S)$ and $V_1(S \setminus H)$, where $e \cap \text{VR}_{k-1}(H, S) \neq \emptyset$.)

**Exercise 23: Numbers of vertices, edges, and faces of $V_k(S)$ (12 points)**

Let $S$ be a set of $n$ point sites in the Euclidean plane satisfying a general position assumption that no three sites are on the same line and no four sites are on the same circle. For $1 \leq i \leq n-1$, let $N_i$, $E_i$, $I_i$, $\mathcal{B}_k$, $\mathcal{S}_i$ be the numbers of faces, edges, vertices, bounded regions, and unbounded faces of $V_i(S)$, respectively, and let $\mathcal{S}_0$ be 0. Please prove the following:

1. $E_k = 3(N_k - 1) - \mathcal{S}_k$ and $I_k = 2(N_k - 1) - \mathcal{S}_k$. (Hint: Euler formula. Due the general position assumption, the degree of a Voronoi vertex is 3).

2. $N_1 = n$, and $N_2 = 3(n-1) - \mathcal{S}_1$, and $N_k = 3(N_{k-1} - 1) - \mathcal{S}_{k-1} - 2\sum_{i=1}^{k-2}(-1)^{k-2-i}(2(N_i - 1) - \mathcal{S}_i)$ implies

$$N_k = 2k(n-k) + k^2 - n + 1 - \sum_{i=0}^{k-1} S_i.$$

   (Hint: By induction on $k$)

3. $\sum_{k=1}^{n-1} \mathcal{B}_k = \binom{n-1}{3}$ (Hint: $\sum_{k=1}^{n-1} I_k = 2\binom{n}{3}$ and $\sum_{k=1}^{n-1} \mathcal{S}_k = 2\binom{n}{2}$)

4. Let $I'_k$ be the number of new vertices of $V_k(S)$. Prove that $I'_k = 2k(n-k) + (k+1)^2 - \sum_{i=1}^{k} \mathcal{S}_i$. (Hint: $N_{k+2} = E_{k+1} - 2I'_k$.)

**Exercise 24: Relation between $V_i(S)$ and $V_{i+1}(S)$ (4 points)**

Assume $\mathrm{VR}_i(H,S)$ has $m$ adjacent regions $\mathrm{VR}_i(H_j,S)$, $1 \le j \le m$. Let $Q$ be $\bigcup_{1\le j\le m} H_j \setminus H$. Prove that $V_{i+1}(S) \cap \mathrm{VR}_i(H,S) = V_1(Q) \cap \mathrm{VR}_i(H,S)$. (Hint: prove that for all site $r \in (S\setminus H)\setminus Q$, $\mathrm{VR}_1(r, S\setminus H) \cap \mathrm{VR}_k(H,S) = \emptyset$. You can first assume the contrary that $\exists r \in (S\setminus H)\setminus Q$ $\mathrm{VR}_1(r, S\setminus H) \cap \mathrm{VR}_k(H,S) \neq \emptyset$, and then show that it will lead to a contradiction. For any point $x \in \mathrm{VR}_1(r, S\setminus H) \cap \mathrm{VR}_k(H,S)$, $\overline{rx}$ will intersect a Voronoi edge $e$ between $\mathrm{VR}_i(H,S)$ and $\mathrm{VR}_i(H_j,S)$ for some $j \in \{1,\ldots,m\}$. Let $y$ be the intersection point between $\overline{rx}$ and $e$. Discuss nearest neighbors of $y$, which will lead to a contradiction from the viewpoint of $e$ and the viewpoint of $\mathrm{VR}_1(r, S\setminus H)$.)