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Probabilistic Analysis
of Algorithms
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Problem Set 4

Problem 1

Let $G = (V, E)$ be a graph with edge lengths $\ell : E \to \mathbb{R}_{>0}$ and edge costs $c : E \to \mathbb{R}_{>0}$. We want to find a path with minimum length as well as minimum total costs. In general there is no such path that optimizes both criteria simultaneously and we are interested in the set of Pareto-optimal paths. Give an algorithm to find the set of Pareto-optimal paths and analyze its worst-case and smoothed running time.

Problem 2

Consider the same setting as in the previous problem, except that we are now interested in the TSP. That is, we assume that G is a complete graph and we want to find a Hamiltonian cycle that is as short and as cheap as possible. Can the set of Pareto-optimal Hamiltonian cycles be computed efficiently (i.e., in polynomial time with respect to the input size and the size of the Pareto set)?

Problem 3

Let E be a set of n elements and F be a family of subsets of E (i.e., F is a subset of the power set of E). Suppose each element $x \in E$ is independently assigned a weight $w(x)$ uniformly at random from the set $\{1, \dots, N\}$. Let the weight of a set $S \in F$ be defined as

$$w(S) := \sum_{x \in S} w(x).$$

Let S^* be one set of F with maximum weight, i.e., there is no other set in F whose weight is higher than $w(S^*)$. Prove the following statement:

$$\mathbf{Pr}\left[\exists S' \in F \backslash \{S^*\} : w(S') = w(S^*)\right] < \frac{n}{N}.$$

In other words: the probability of S^* to be a unique maximum is at least $1 - \frac{n}{N}$.

Problem 4

In a quiz show three participants can win a trip to Hawaii if they win the following game: Each participant gets independently and uniformly at random either a red or a green hat; he cannot see the color of his hat but he sees the colors of the others. Then, without communicating, all three write down either "red", "green" or "unknown". The three players win if at least one player wrote down "red" or "green" and if all players that wrote down "red" or "green" correctly guessed the color of their own hat.

(a) Give a strategy for the three players with a winning probability of exactly 50%.

(b) Is there a strategy that guarantees a winning probability of more than 50%?