Probabilistic Analysis of Algorithms

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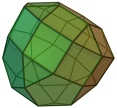
Discrete Optimization

Many problems and algorithms seem well understood.



Linear Programming
efficient algorithms (ellipsoid, interior point)

Knapsack Problem (KP)
NP-hard, FPTAS exists

Traveling Salesperson Problem (TSP)
NP-hard, even hard to approximate

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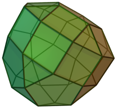

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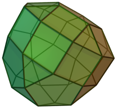

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⇒ big gap between theory and practice

Outline

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1. **Linear Programming**
   Why is the simplex method usually efficient?
   Smoothed Analysis – analysis of algorithms beyond worst case
2. **Traveling Salesperson Problem**
   Why is local search successful?
3. **Smoothed Analysis**
   Overview of known results

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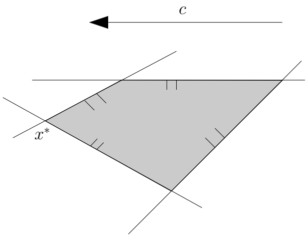
Linear Programs (LPs)

- variables: $x_1, \ldots, x_n \in \mathbb{R}$
- linear objective function:
  $\max c^T x = c_1 x_1 + \ldots + c_n x_n$
- m linear constraints:

$$\begin{array}{ccc} a_{1,1}x_1 & + \ldots + & a_{1,n}x_n \le b_1 \\ & \vdots & \\ a_{m,1}x_1 & + \ldots + & a_{m,n}x_n \le b_m \end{array}$$



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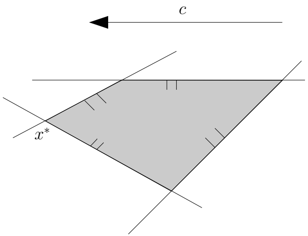
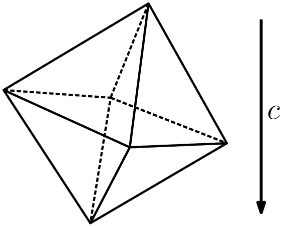


Complexity of LPs

LPs can be solved in **polynomial time** by the ellipsoid method [Khachiyan 1979] and the interior point method [Karmarkar 1984].

Simplex Algorithm



Simplex Algorithm

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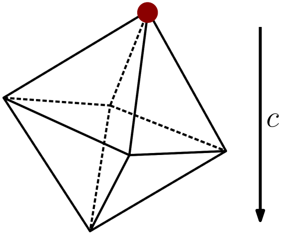

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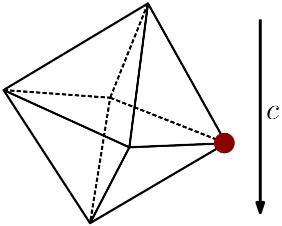

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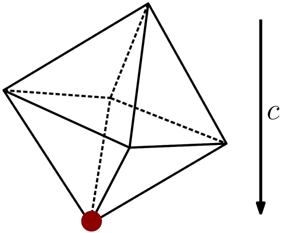

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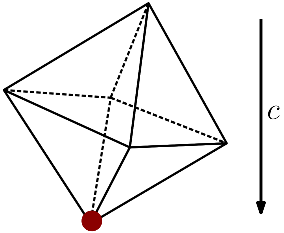

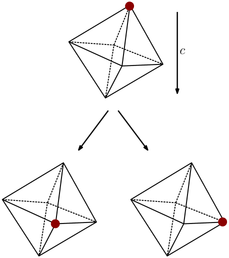

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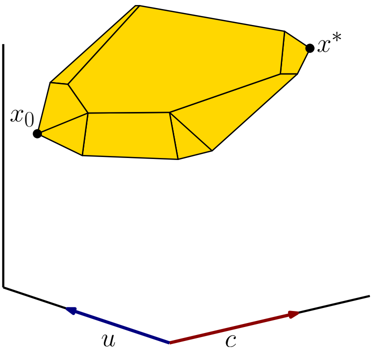
Pivot Rules

- Which vertex is chosen if there are multiple options?
- Different **pivot rules** suggested: random, steepest descent, shadow vertex pivot rule, ...

Shadow Vertex Pivot Rule

- Let x_0 be some vertex of the polytope.
- Compute $u \in \mathbb{R}^d$ such that x_0 maximizes $u^T x$.



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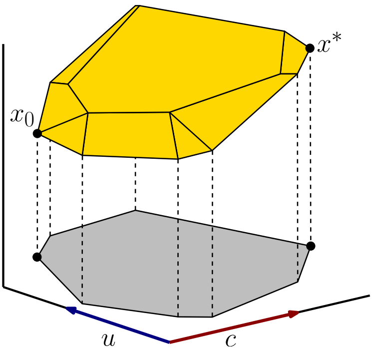

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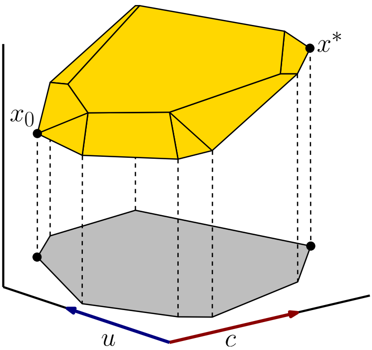

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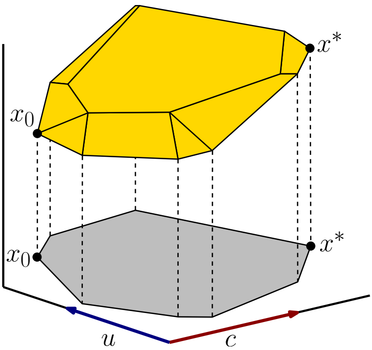

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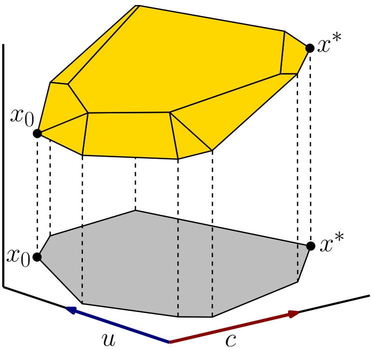

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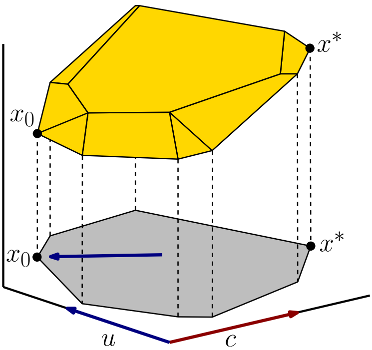

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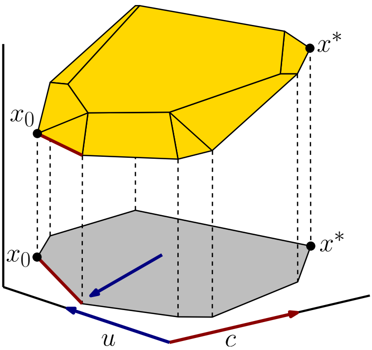

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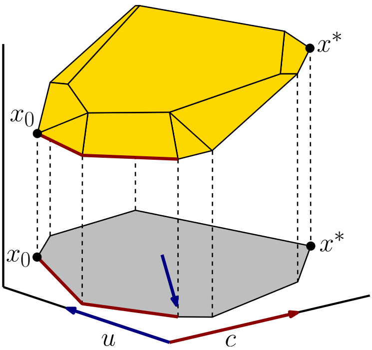

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Simplex Algorithm – Shadow Vertex Pivot Rule

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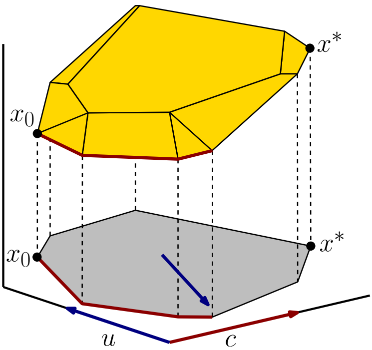

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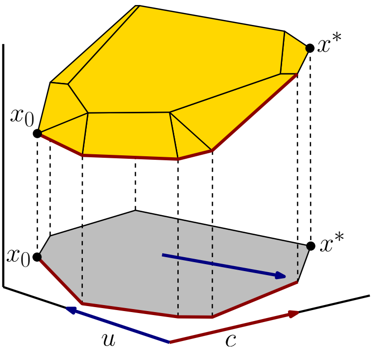

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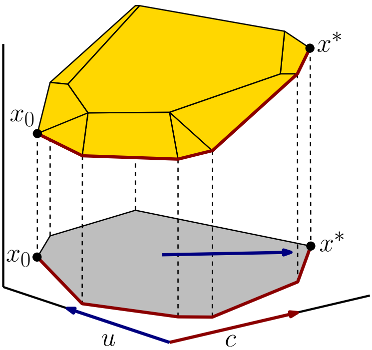

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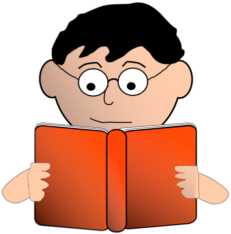

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- shadow vertex pivot rule requires exponential number of steps
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- ellipsoid and interior point methods are efficient

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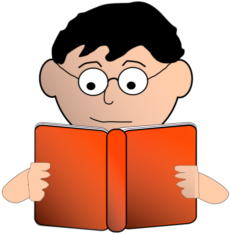

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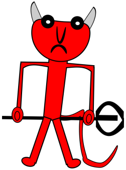
Engineers say...

- simplex method usually fastest algorithm in practice
- requires usually only $\Theta(m)$ steps
- clearly outperforms ellipsoid method

Reason for Gap between Theory and Practice

- Worst-case complexity is too pessimistic!
- There are (artificial) worst-case LPs on which the simplex method is not efficient. These LPs, however, do not occur in practice.
  e.g., $a_{1,i} = 2^i$, $\sum_i a_{2,i} \equiv 3 \bmod 5$, …

Adversary



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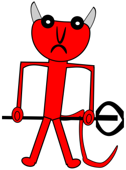

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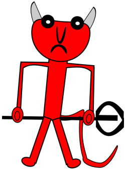

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Goal

Find a more realistic performance measure that is not just based on the worst case.

Smoothed Analysis

Observation: In worst-case analysis, the adversary is too powerful.

Idea: Let's weaken him!

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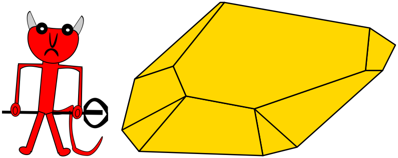
Perturbed LPs

- Step 1: Adversary specifies arbitrary LP:
  $\max c^T x$ subject to $a_1^T x \leq b_1 \ldots a_n^T x \leq b_n$.
  W. l. o. g. $\|(a_i, b_i)\| = 1$.



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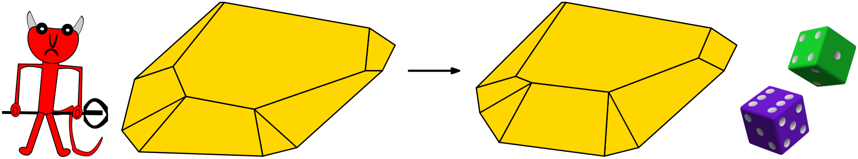

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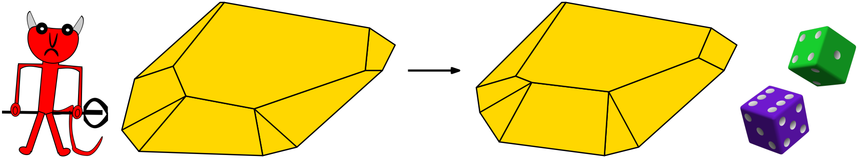
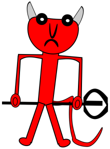


Smoothed Running Time

$=$ worst expected running time the adversary can achieve



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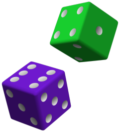
Adversary
chooses input I

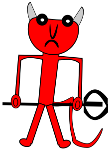


Step 2: Random
perturbation
$I \rightarrow \mathrm{per}_\sigma(I)$

Formal Definition:

$\mathrm{LP}(n, m) =$ set of LPs with n variables and m constraints

$T(I) =$ number of steps of simplex method on LP I



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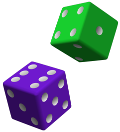

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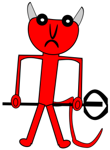
smoothed run time $T^{\text{smooth}}(n, m, \sigma) = \max_{I \in \mathrm{LP}(n,m)} \mathbf{E}[T(\mathrm{per}_\sigma(I))]$



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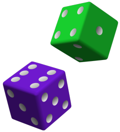

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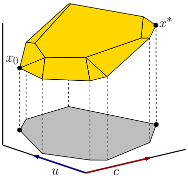
Why do we consider this model?

- First step models unknown structure of the input.
- Second step models random influences, e.g., measurement errors, numerical imprecision, rounding, ...
- smoothed running time low $\Rightarrow$ bad instances are unlikely to occur
- σ determines the amount of randomness

Lemma [Spielman and Teng (STOC 2001)]

For every fixed plane and every LP the adversary can choose, after the perturbation, the expected number of edges on the shadow is

$$O\left(\mathrm{poly}\left(n, m, \sigma^{-1}\right)\right).$$



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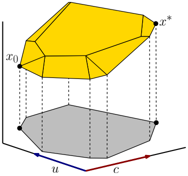

Theorem [Spielman and Teng (STOC 2001)]

The smoothed running time of the simplex algorithm with shadow vertex pivot rule is $O\left(\mathrm{poly}\left(n, m, \sigma^{-1}\right)\right)$.

Already for small perturbations exponential running time is unlikely.

Smoothed Analysis of the Simplex Algorithm

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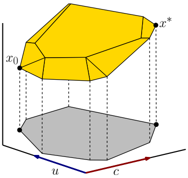

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Main Difficulties in Proof of Theorem:

- x_0 is found in phase I $\rightarrow$ no Gaussian distribution of coefficients
- In phase II, the plane onto which the polytope is projected is not independent of the perturbations.

Improved Analysis

Theorem [Vershynin (FOCS 2006)]

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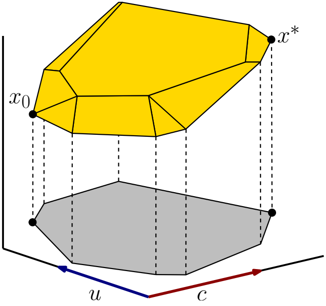

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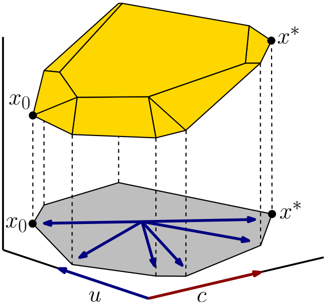

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  $\Rightarrow$ The plane is not correlated with the perturbed polytope.
- With high prob. no angle between consecutive vertices is too small.

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1. **Linear Programming**
   Why is the simplex method usually efficient?
   smoothed analysis – analysis of algorithms beyond worst case
2. **Traveling Salesperson Problem**
   Why is local search successful?
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Traveling Salesperson Problem (TSP)

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Metric TSP: APX-hard **Euclidean TSP:** PTAS exists

Numerous Experimental Studies:
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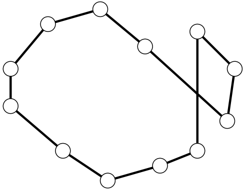

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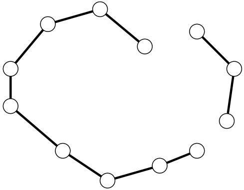

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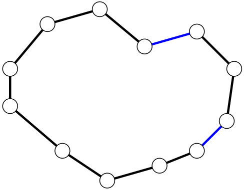

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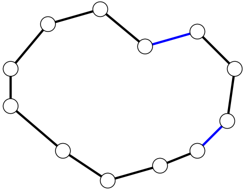

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- approximation ratio: ≈ 1.05
  number of steps: $\leq n \cdot \log n$

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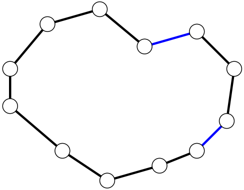

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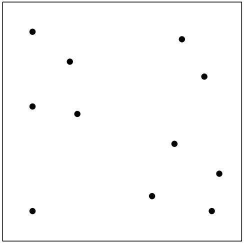

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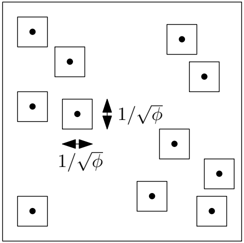

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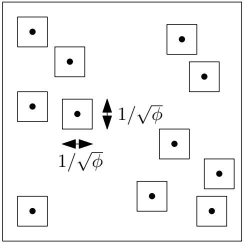

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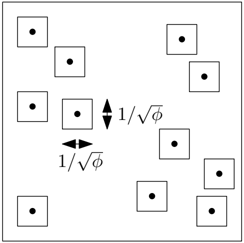

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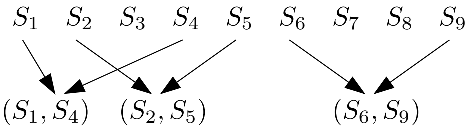

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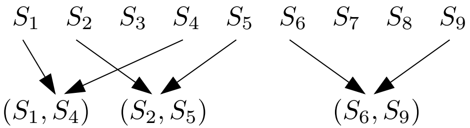

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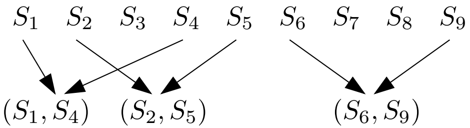

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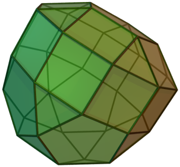

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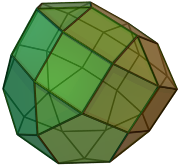
Overview of Results on Smoothed Analsyis



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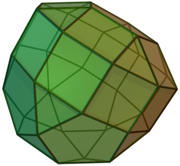

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Combinatorial Optimization

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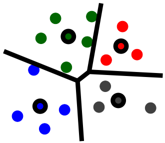

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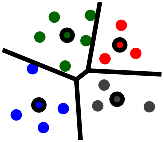

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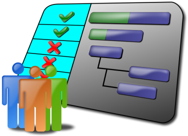
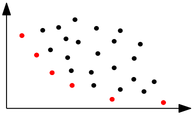


Scheduling

Multilevel Feedback Algo [Becchetti, Leonardi, Marchetti-Spaccamela, Schäfer, Vredeveld (FOCS 2003)]

Local Search Algos [Brunsch, R., Rutten, Vredeveld (ESA 2011)]



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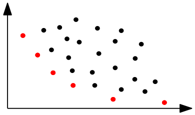

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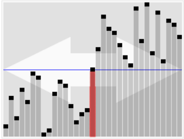

Classical Algorithms and Data Structures

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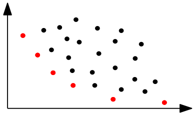

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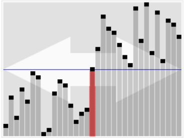

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