4. Configuration Space

A configuration σ over N

- N is a set of objects, e.g., points, line segments, half-spaces, etc.
- σ is a pair $(D, L) = (D(\sigma), L(\sigma))$, where D and L are disjoint subsets of N.
  - The objects in D are called the *triggers* associated with σ. D are also called the objects that *define* σ, and σ is called *adjacent* to the objects in D.
  - The objects in L are called the *stoppers* associated with σ. They are also called the objects that *conflict* with σ.
  - The *degree* $d(\sigma)$ is defined to be the cardinality of $D(\sigma)$.
  - The *level* or the *conflict size* $l(\sigma)$ is defined to be the cardinality of $L(\sigma)$.
  - Sometimes, "d" and "l" stand for degree and level respectively.

Bounded degree property

The degree of each configuration is bounded by a constant

Configuration space

A configuration space $\Pi(N)$ over the universe N is just a set of configurations over N with the bounded degree property.

- $\Pi(N)$ is a multi-set. That is, several "distinct" configurations in $\Pi(N)$ can have the same trigger and stopper set.
- The *size* $\pi(N)$ or $|\Pi(N)|$ of $\Pi(N)$ means the total number of distinct configurations in $\Pi(N)$. Here the distinct configurations having the same trigger and stopper set are counted separately.
- The *reduced size* $\tilde{\pi}(N)$ of $\Pi(N)$ is the total number of configurations in $\Pi(N)$ when the configurations with the same trigger and stopper set are not counted separately.

Levels and Active

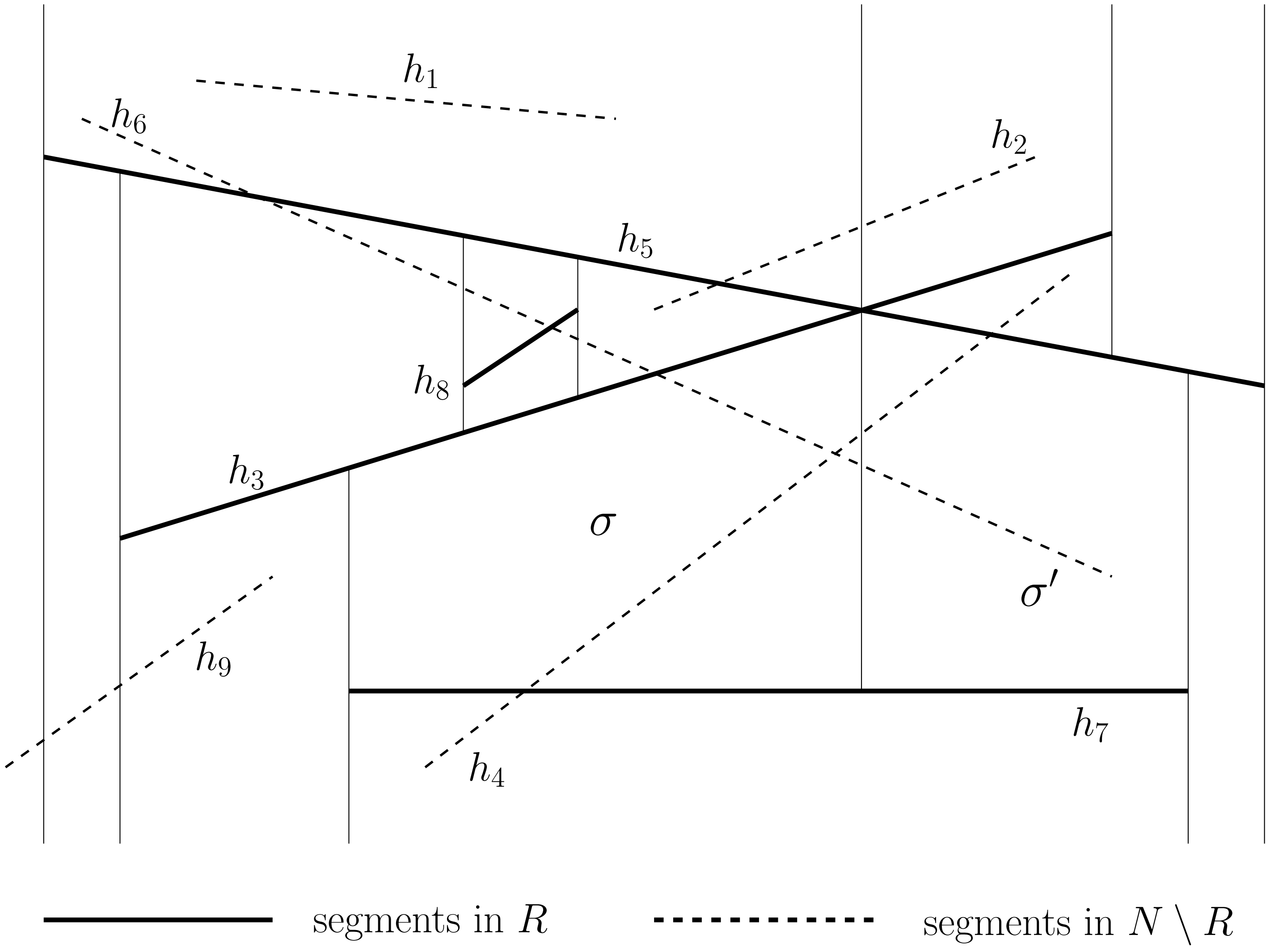
For each integer $i \geq 0$, $\Pi^i(N)$ is defined to be the set of configurations in $\Pi(N)$ with level i.

- $\pi^i(N)$ denotes the size of $\Pi^i(N)$.
- The configurations in $\Pi^0(N)$ are said to be *active* over N.
- The configurations in $\Pi^i(N)$ are said to be *partially active* over N with level i.

Example

Trapezoidal Decomposition

- Let N be a set of segments in the plane.
- A trapezoid σ in the plane is *feasible* over N is σ occurs in the trapezoidal decomposition $H(R)$, for some subset $R \subseteq N$.
- Each feasible trapezoid can arise in the incremental construction, and each trapezoid that arise in the incremental construction is feasible.
- For a feasible trapezoid σ
  - $D(\sigma)$ is the set of segments in N that are adjacent to the boundary of σ
  - $L(\sigma)$ is the set of segments in N that intersect the interior of σ.
- General Position Assumption
  - No three segments in N intersect at the same point
  - No two endpoints of segments in N share the same x-coordinate
  - No two intersection points among segments share the same x-coordinate
- Under the general position assumption, $d(\sigma)$ is at most 4.



Illustration

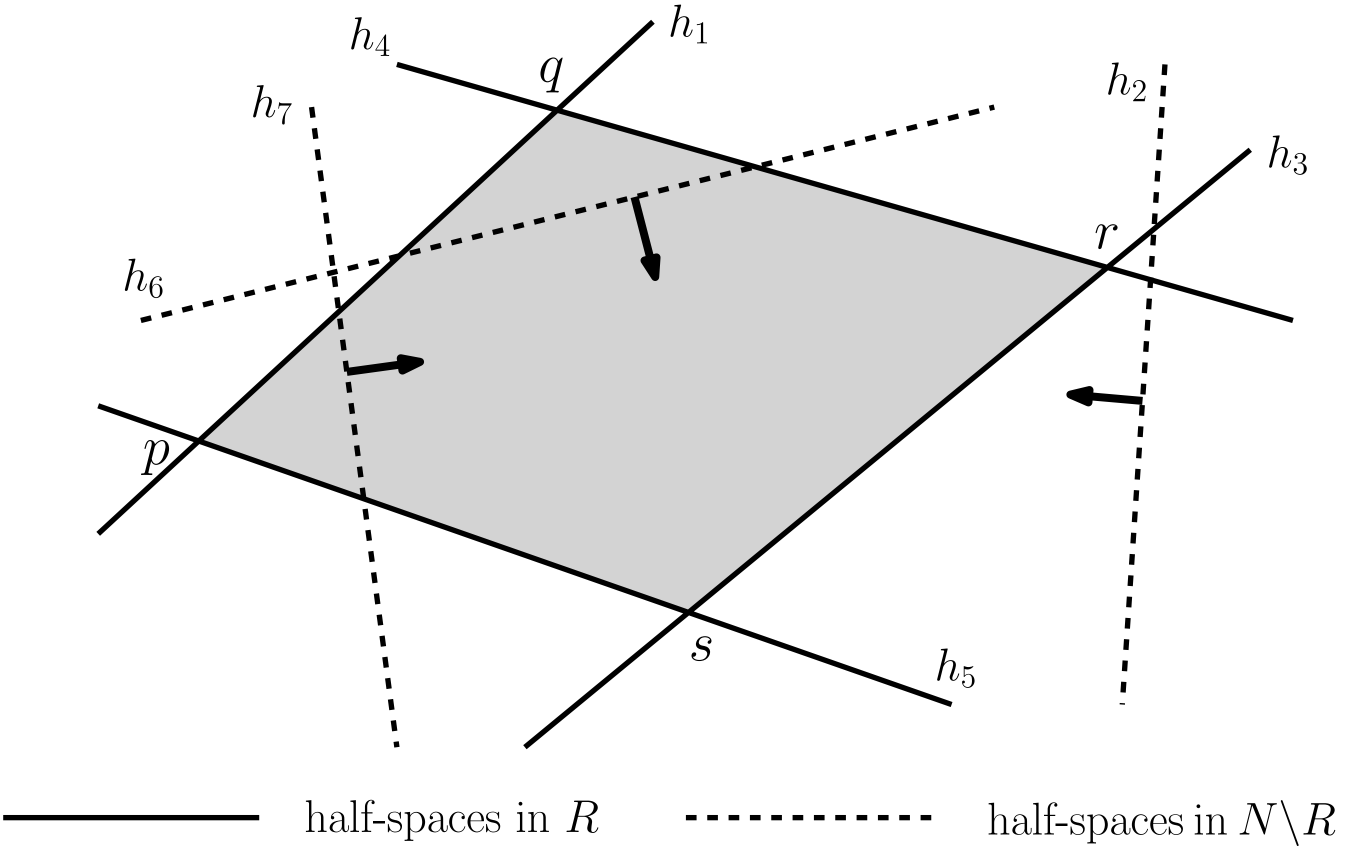
- $N = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\}$
- $R = \{h_3, h_5, h_7, h_8\}$
- Trapezoid σ
  - $D(\sigma) = \{h_3, h_5, h_7\}$
  - $L(\sigma) = \{h_4, h_6\}$
- σ and σ' share the same trigger and stopper set.

Subspace $\Pi(R)$ **over** N

- R is a subset of N
- Given a configuration $\sigma \in \Pi(N)$ such that $D(\sigma) \subseteq R$, the *restriction* $\sigma \downarrow R$ is the configuration over the universe R whose trigger set is $D(\sigma)$ and whose conflict set is $L(\sigma) \cap R$.
- $$\Pi(R) = \{\sigma \downarrow R \mid \sigma \in \Pi(N), D(\sigma) \subseteq R\}.$$
- $L(\sigma)$ is the conflict list of σ relative to N
- $L(\sigma) \cap R$ is the conflict list of σ relative to R
- σ is *active* over R if $D(\sigma) \subseteq R$ and $L(\sigma) \cap R = \emptyset$
- $\Pi^0(R)$ is the set of configurations in $\Pi(N)$ that are active over R.
- Without considering D and L, $H(N) = \Pi^0(N)$ and $H(R) = \Pi^0(R)$.

Example **Convex Polytope**

- N is a set of half-spaces in the d-dimensional space
- The *convex polytope* $G(N)$ of N is the common intersection among the half-spaces in N.
- A configuration σ over N is a vertex of $G(N)$
  - $D(\sigma)$ is the set of half-spaces in N whose defining hyperplane contains σ
  - $L(\sigma)$ is the set of half-spaces in N which **Do Not** contains σ
- General Position Assumption: For $i \leq d$, the intersection among $i + 1$ hyperplanes which define half-spaces in N is $(d - i)$-dimensional.
  - When $d = 3$, the intersection between two planes is a line, the intersection among three planes is a point, and the intersection among more than three planes is empty.
- Under the general position assumption, $D(\sigma)$ is exactly d.
- $G(N)$ can be unbounded. In this situation, the unbounded endpoint of an unbounded edge of $G(N)$ is also viewed as a vertex σ of $G(N)$, and $D(\sigma) = d - 1$



Illustration

- $N = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$
- $R = \{h_1, h_3, h_4, h_5\}$
- p, q, r, and s form $G(R)$
- $D(q) = \{h_1, h_4\}$, $L(q) = \{h_6\}$
- $D(p) = \{h_1, h_5\}$, $L(p) = \{h_6, h_7\}$

Randomized Incremental Construction

1. Generate a random sequence (permutation) S_1, S_2, ..., S_n of the objects in N
2. Add the objects in N one at a time according to this sequence.
   - Let N^i denote the set of the first i added objects
   - At the i^{th} stage, the algorithm maintains the set $\Pi^0(N^i)$ of active configurations
   - For each configuration $\sigma \in \Pi^0(N^i)$, we also maintain its conflict set $L(\sigma)$ relative to N, and its size $l(\sigma)$ is the conflict size of σ relative to N.

Structurel and Conflict Changes

- The *structural change* at time i is the number of newly created and destroyed configurations at time i
  - Destroyed configurations at time i are the configurations that were present in $\Pi^0(N^{i-1})$ but not in $\Pi^0(N^i)$
  - Newly created configurations at time i are the configurations that were present in $\Pi^0(N^i)$ but not in $\Pi^0(N^{i-1})$.
- The *conflict change* at time i is the sum of conflict sizes of the newly created and destroyed configurations at time i
- The total structural change during the preceding incremental construction is the sum of the structural changes at time i, over all i.
- The total conflict change is defined similarly.

Important Quantities

- For each $j \leq n$, $e(j)$ denotes the expected size of $\Pi^0(N^j)$, where N^j is a random sample of N of size j.
  - Each element in N is equally likely to be present in N^j.
- $d = d(\Pi)$ denotes the maximum degree of a configuration in $\Pi(N)$.
  - By the definition of a configuration space, d is bounded by a constant.

Theorem

The expected value of the total structural change in the randomized incremental construction is bounded by

$$\sum_{j=1}^{n} d \cdot e(j)/j.$$

proof

- Since each destroyed configuration must be constrcuted before, it is sufficient to count newly created configurations at time j, for all j
- Let S_j be the j^{th} inserted object.
- Each object in N^j is equally likely to be S_j.
- Since each configuration is defined by at most d objects and $\Pi^0(N_j)$ has at most $e(j)$ configurations, the expected number of configuration in $\Pi^0(N_j)$ defined by S_j is bounded by $d \cdot e(j)/j$.

Theorem

The expected value of the total conflict change in the randomized incremental construction is bounded by

$$\sum_{j=1}^{n} d^2(n-j)e(j)/j^2.$$

Intuital Idea

- The conflict size for a configuration in $\Pi^0(N^1)$ is $O(n)$.
- The conflict size for a configuration in $\Pi^0(N^n)$ is 0.
- The epxcted conflict size for a configuration in $\Pi^0(N^j)$ would probably be $O((n-j)/j)$.

proof

- Let S_j and S_{j+1} be the j^{th} and $(j+1)^{\text{st}}$ inserted object.
- For an object $S \in N^j$ and an object $I \in N \setminus N^j$, $k(N^j, S, I)$ is the number of configurations in $\Pi^0(N^j)$ defined by S and conflicted by I, and $k(N^j, I)$ is the number of configurations in $\Pi^0(N^j)$ conflicted by I.
- Since each object in N^j is equally likely to be S_j and each configuration in $\Pi^0(N^j)$ is defined by at most d objects, the expected value of $k(N^j, S_j, I)$ is proportional to

$$\frac{1}{j}\sum_{S\in N^j} k(N^j, S, I) \leq \frac{d \cdot k(N^j, I)}{j}$$

- Summing over I, the expected total conflict size of the newly created configuration during the j^{th} addition is proportional to

$$\frac{d}{j}\sum_{I\in N\setminus N^j} k(N^j, I) \tag{1}$$

- Since each object in $N \setminus N^j$ is equally likely to be S_{j+1},

$$E[k(N^j, S_{j+1})] = \frac{1}{n-j}\sum_{I\in N\setminus N^j} k(N^j, I).$$

- Equation (1) can be re-written as

$$d\frac{n-j}{j}E[k(N^j, S_{j+1})].$$

- Since $k(N^j, S_{j+1})$ is also the number of configurations in $\Pi^0(N^j)$ that are destroyed due to the insertion of S_{j+1}, the expected total conflict size of all configuration during the incremental construction is bounded by

$$\sum_{j=1}^{n} d\frac{n-j}{j} \times \text{ expected number of configurations destroyed at time } j+1 \tag{2}$$

proof (continue)

- By linearity of expectation, (2) is the same the expected value of

$$\sum_{\sigma} d\frac{n-[j(\sigma)-1]}{j(\sigma)-1}, \tag{3}$$

  where σ ranges over all configuration created during the incremental construction, and $j(\sigma)$ is the time when σ is destroyed.

- Let $i(\sigma)$ be the time when σ is created. Since $i(\sigma) \le j(\sigma)-1$,

$$\frac{n-[j(\sigma)-1]}{j(\sigma)-1} = \frac{n}{j(\sigma)-1} - 1 \le \frac{n}{i(\sigma)} - 1 = \frac{n-i(\sigma)}{i(\sigma)}.$$

- (3) can be re-arranged as

$$\sum_{j=1}^{n} d\frac{n-j}{j} \times \#\text{ of configurations created at time } j \tag{4}$$

- The expected total conflict change is bounded by

$$\sum_{j=1}^{n} d\frac{n-j}{j} \times \text{expected number of configurations created at time } j \tag{5}$$

- Since the expected number of configurations created at time j is bounded by $\frac{d}{j}e(j)$, the equation in (5) becomes

$$\sum_{j=1}^{n} d^2\frac{n-j}{j}\frac{e(j)}{j}. \tag{3}$$

Trivial configuration space

A configuration space $\Pi(N)$ over a given object set N is *trivial* if for every subset $M \subseteq N$, the number of configurations in $\Pi(N)$ active over M is $O(1)$.

Example

- N is a set of n line segments
- $\Pi(N)$ is the configurations space of feasible trapezoids over N
- For every fixed point p in the plane, $\Pi_p(N)$ is the subsppace of all feasible trapezoids in $\Pi(N)$ that contains p.
- Since for any $M \subseteq N$ exactly one trapezoid in $H(M)$ can contains p, $\Pi_p(N)$ is a trivail configuration space
- The expected structural change becomes $\sum_{j=1}^{n} dO(1/j) = O(\log n)$.
- It is the expected length of a search path in the history graph for a point location query.