

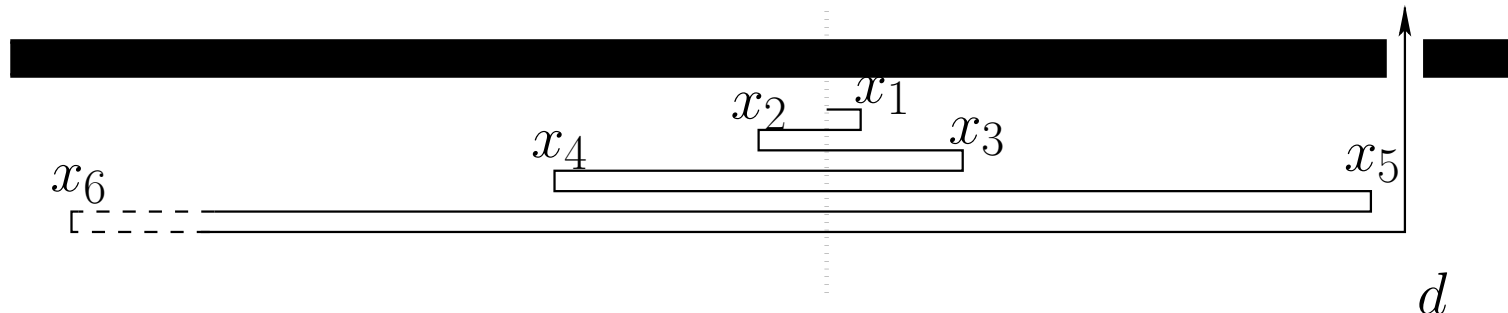
Online Motion Planning MA-INF 1314

Searching Points/Rays

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Rep.: Searching for a point!

- 2-ray search: Point on a line
- Compare with shortest path, competitive?
- Reasonable strategy: Depth x_1 , depth x_2 and so on
- Target at least step 1 away!
- Worst-Case, just behind d , one add. turn!
- Strategy, such that: $\sum_{i=0}^{k+1} 2x_i + x_k \leq Cx_k$
- Minimize: $\frac{\sum_{i=0}^{k+1} x_i}{x_k}$, Functional!



Rep.: Theorem Gal 1980

If functional F_k fulfils:

- i) F_k continuous
- ii) F_k unimodal: $F_k(A \cdot X) = F_k(X)$ und
 $F_k(X + Y) \leq \max\{F_k(X), F_k(Y)\}$,
- iii) $\liminf_{a \mapsto \infty} F_k \left(\frac{1}{a^{k+i}}, \frac{1}{a^{k+i-1}}, \dots, \frac{1}{a}, 1 \right) =$
 $\liminf_{\epsilon_{k+i}, \epsilon_{k+i-1}, \dots, \epsilon_1 \mapsto 0} F_k \left(\epsilon_{k+i}, \epsilon_{k+i-1}, \dots, \epsilon_1, 1 \right)$,
- iv) $\liminf_{a \mapsto 0} F_k \left(1, a, a^2, \dots, a^{k+i} \right) =$
 $\liminf_{\epsilon_{k+i}, \epsilon_{k+i-1}, \dots, \epsilon_1 \mapsto 0} F_k \left(1, \epsilon_1, \epsilon_2, \dots, \epsilon_{k+i} \right)$,
- v) $F_{k+1}(f_1, \dots, f_{k+i+1}) \geq F_k(f_2, \dots, f_{k+i+1})$.

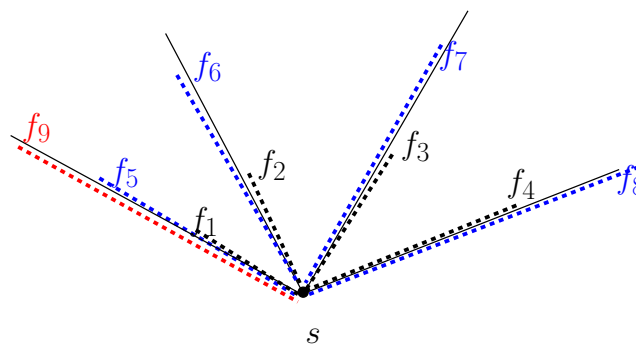
Then: $\sup_k F_k(X) \geq \inf_a \sup_k F_k(A_a)$ mit $A_a = a^0, a^1, a^2, \dots$ und $a > 1$.

Rep.: Example 2-ray search

- $F_k(f_1, f_2, \dots) := \frac{\sum_{i=1}^{k+1} f_i}{f_k}$ for all k .■
- Unimodal $F_k(A \cdot X) = F_k(X)$ and $F_k(X + Y) \leq \max\{F_k(X), F_k(Y)\}$?■
- $\frac{\sum_{i=1}^{k+1} A \cdot f_i}{A \cdot f_k} = \frac{\sum_{i=1}^{k+1} f_i}{f_k}$
- $F_k(X + Y) \leq \max\{F_k(X), F_k(Y)\}$?■
- Follows from $\frac{a}{b} \geq \frac{c}{d} \Leftrightarrow \frac{a+c}{d+b} \leq \frac{a}{b}$ ■
- Simple equivalence!■
- Optimize: $f_k(a) := \frac{\sum_{i=1}^{k+1} a^i}{a^k}$ ■
- Minimized by $a = 2$ ■

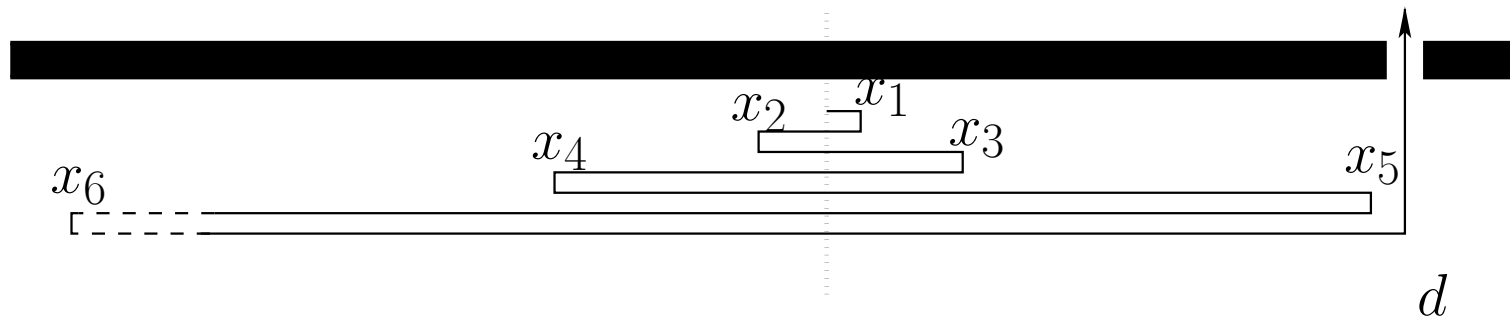
Rep.: Search on m-rays

- **Lemma** For the m-ray search problem there is always an optimal competitive strategy (f_1, f_2, \dots) that visits the rays in a periodic order and with overall increasing depth. ■
- periodic and monotone: (f_j, J_j) , $J_j = j + m$, $f_j \geq f_{j-1}$ ■
- Proof: First index with: $f_j > f_{j+1}$, $J_j > J_{j+1}$, Exchange values and the order on the rays, successively! ■
- (f_j, J_j) , $J_j = j + m$, $f_j \geq f_{j-1}$ Theorem of Gal can be applied! ■



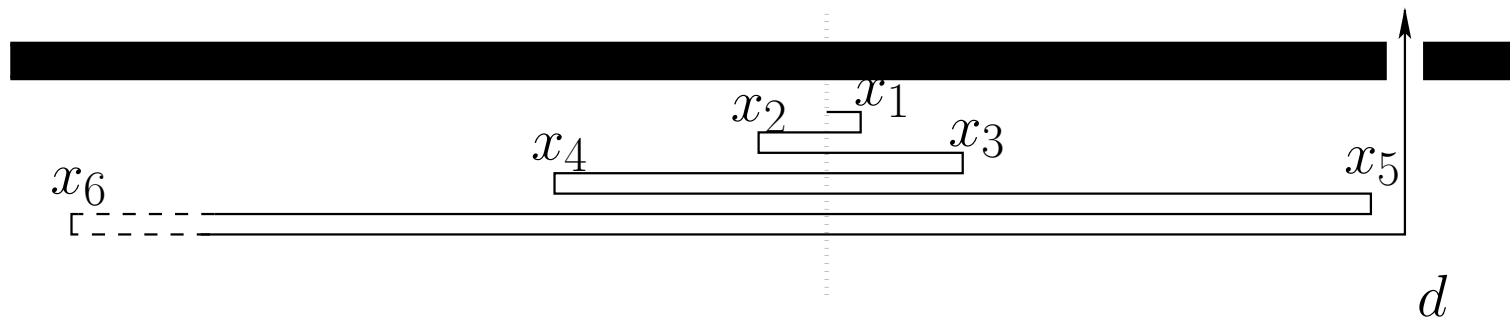
Other approach: Optimality for equations!

- Reasonable strategy, ratio: $\frac{\sum_{i=1}^{k+1} 2x_i + x_k}{x_k} = 1 + 2\frac{\sum_{i=1}^{k+1} x_i}{x_k}$ ■
- Ass.: C optimal, $\frac{\sum_{i=1}^{k+1} x_i}{x_k} \leq \frac{(C-1)}{2}$ ■
- There is strategy $(x'_1, x'_2, x'_3 \dots)$ s. th. $\frac{\sum_{i=1}^{k+1} x'_i}{x'_k} = \frac{(C-1)}{2}$ for all k ■
- Monotonically increasing in x'_j ($j \neq k$), decreasing in x'_k ■
- First k with: $\frac{\sum_{i=1}^{k+1} x_i}{x_k} < \frac{(C-1)}{2}$, decrease x_k ■
- $\frac{\sum_{i=1}^k x_i}{x_{k-1}} < \frac{(C-1)}{2}$!, x_{k-1} decrease etc., monotonically decreasing sequence, bounded, converges! Non-constructive! ■



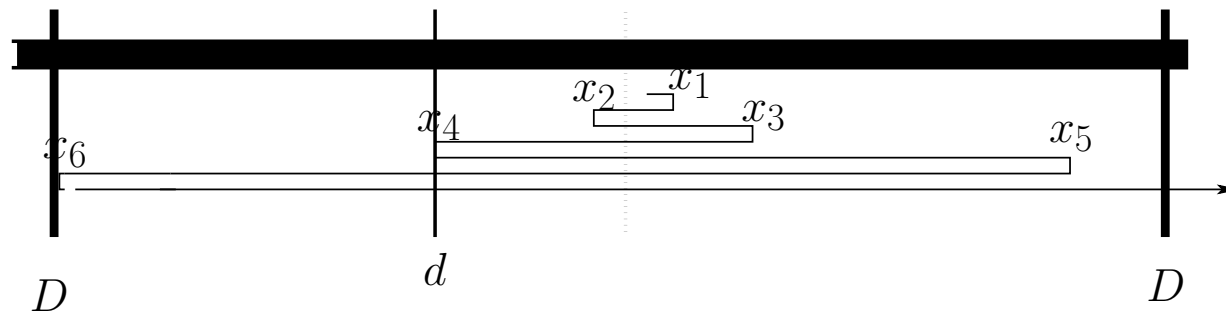
Other approach: Optimality for equations!

- Set: $\frac{\sum_{i=1}^{k+1} x'_i}{x'_k} = \frac{(C-1)}{2}$ for all k ■
- $\sum_{i=1}^{k+1} x'_i - \sum_{i=1}^k x'_i = \frac{(C-1)}{2} (x'_k - x'_{k-1})$ ■
- Thus: $C' (x'_k - x'_{k-1}) = x'_{k+1}$, Recurrence!■
- Solve a recurrence! Analytically! Blackboard!■
- Characteristical polynom: No solution $C' < 4$ ■
- $x'_i = (i + 1)2^i$ with $C' = 4$ is a solution! Blackboard! ■Optimal!■



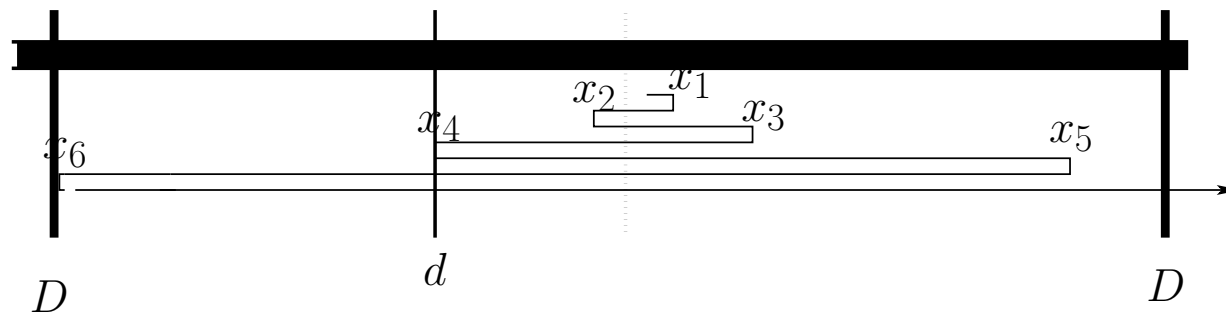
2-ray search, restricted distance

- Assume goal is no more than dist. $\leq D$ away
- Exactly D ! Simple ratio 3!
- Find optimal strategy, minimize C !
- Vice-versa: C is given! Find the largest distance D (reach R) that still allows C competitive search.
- One side with $f_{\text{Ende}} = R$, the other side arbitrarily large!



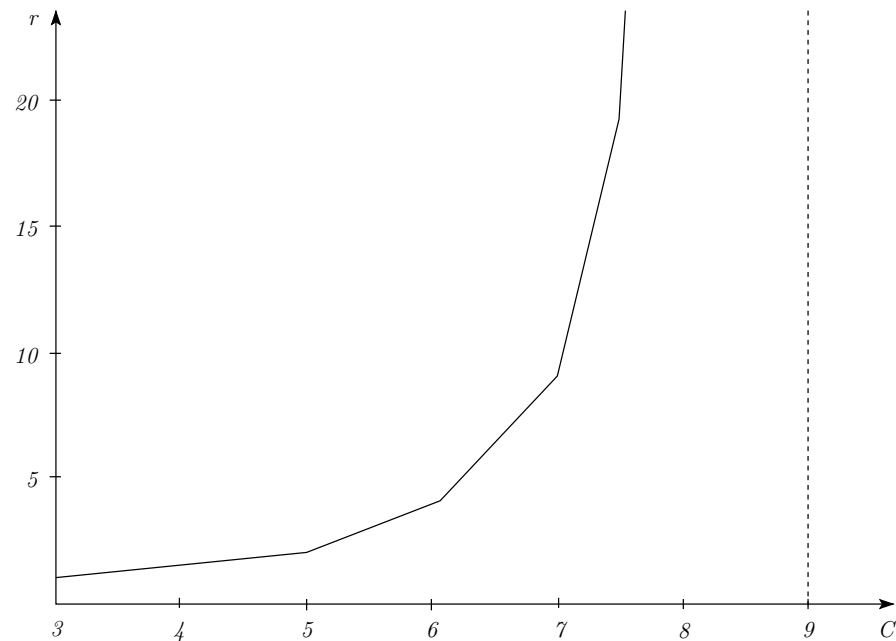
2-ray search, maximal reach R

- C given, optimal reach R ! ■
- **Theorem** The strategy with equality in any step maximizes the reach R ! ■
- Strategy: $\frac{\sum_{i=1}^{k+1} x_i}{x_k} = \frac{(C-1)}{2}$, first step: $x_1 = \frac{(C-1)}{2}$ ■
- Recurrence: $x_0 = 1$, $x_{-1} = 0$, $x_{k+1} = \frac{(C-1)}{2}(x_k - x_{k-1})$ ■
- Strategy is optimal! By means of the Comp. Geom. lecture! ■



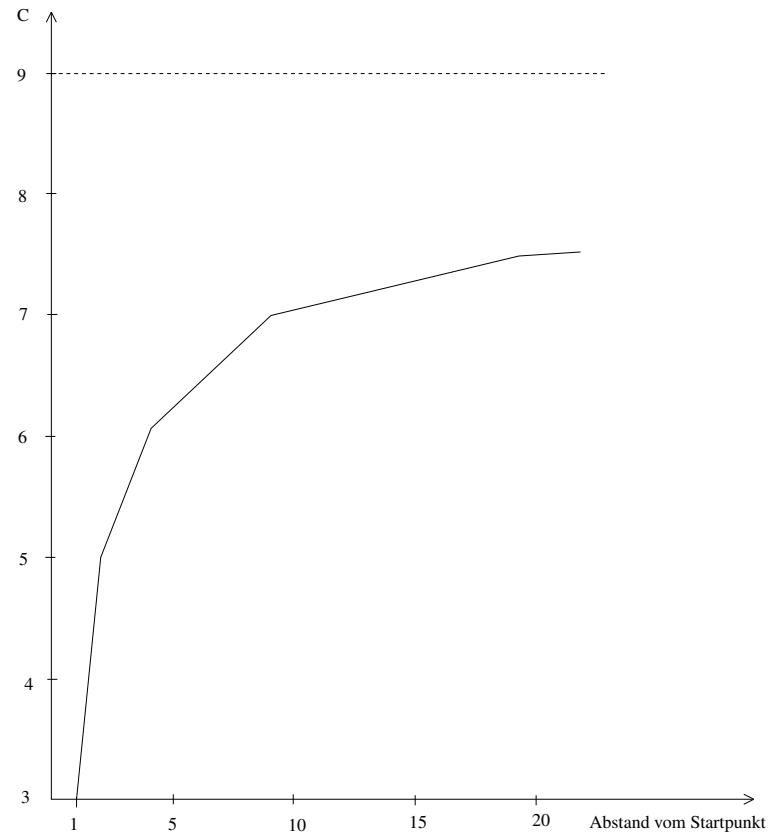
2-ray search, maximal reach R

- $f(C) :=$ maximal reach depending on C
- Bends are more steps!



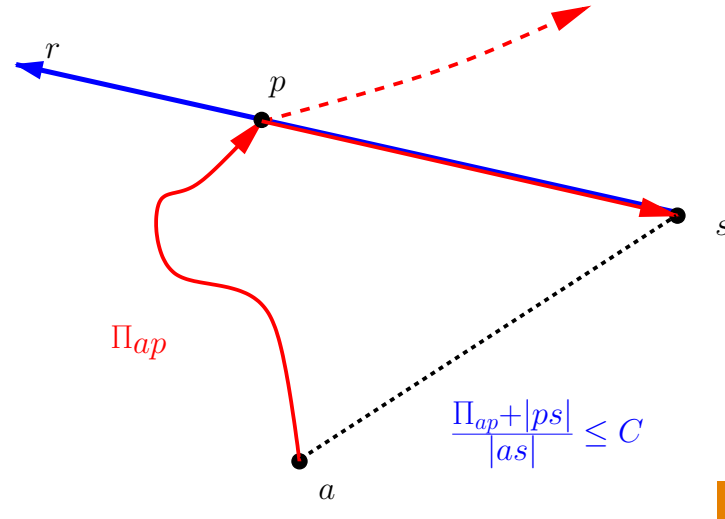
2-ray search, given distance R

- $f(C) :=$ maximal reach depending on C
- Rotate, R given, binary search!



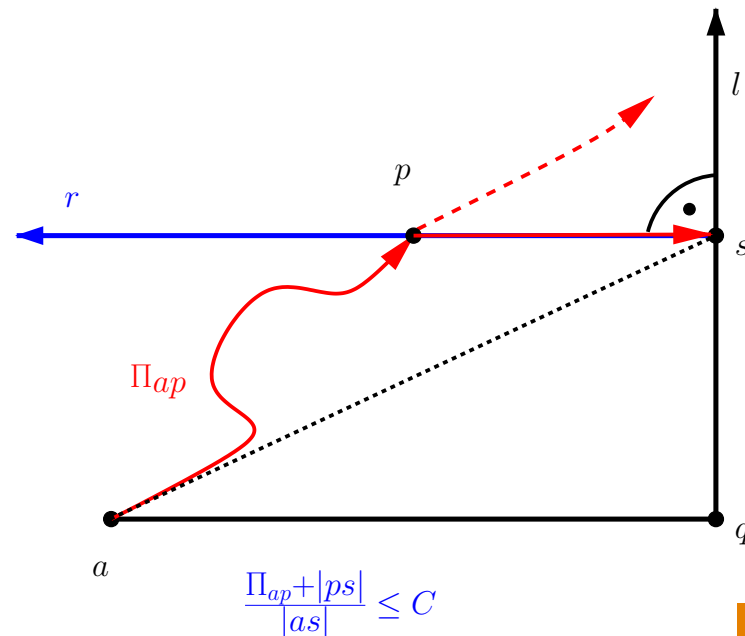
Searching for the origin of ray

- Unknown ray r in the plane, unknown origin s
- Startpoint a
- Searchpath Π , hits r , detects s , move to s
- Shortest path OPT, build the ratio
- Π has *competitive ratio* C if inequality holds for all rays
- Task: Find searchpath Π with the minimal C



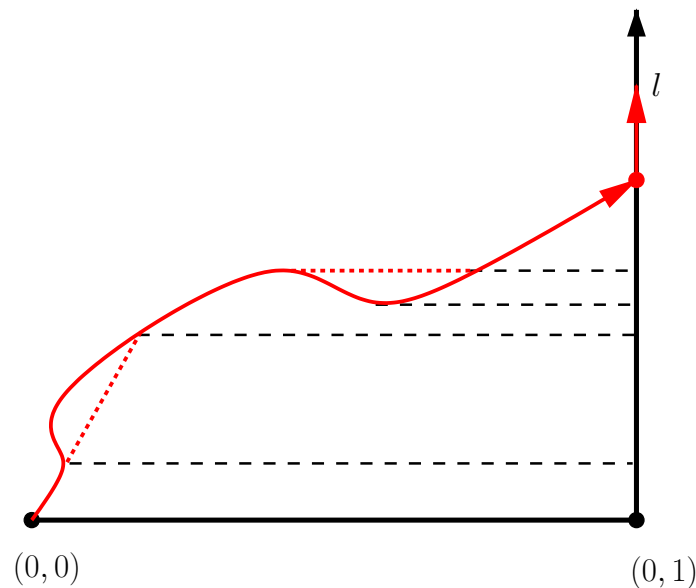
The Window-Shopper-Problem

- Unknown ray starts at s on *known* vertical line l (window)■
- Ray starts perpendicular to l ■
- aq runs parallel to r ■
- *Motivation*: Move along a window until you *detect* an item■
- Move to the item■



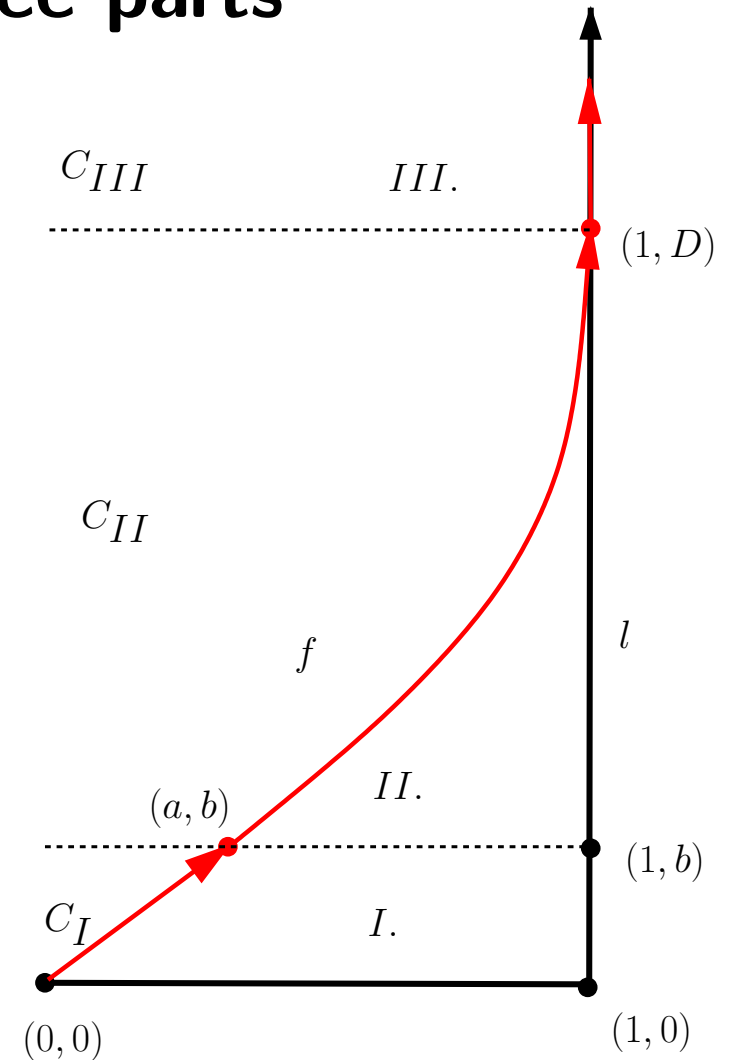
Some observations

- Any reasonable strategy is monotone in x and y
- Otherwise: Optimize for some s on l
- Finally hits the *window*
- Ratio is close to 1 in the beginning, but bigger than 1
- Ratio goes to 1 at the end



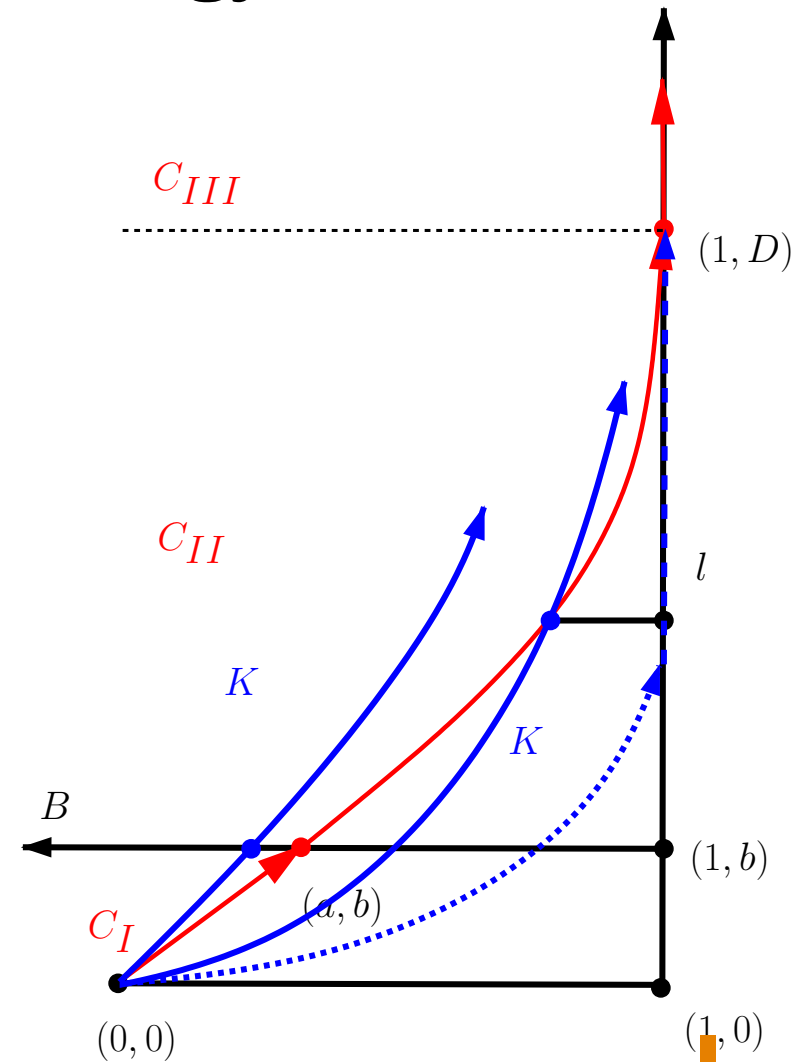
Strategy design: Three parts

- A line segment from $(0, 0)$ to (a, b) with **increasing** ratio for s between $(1, 0)$ and $(1, b)$ ■
- A curve f from (a, b) to some point $(1, D)$ on l which has **the same** ratio for s between $(1, b)$ and $(1, D)$ ■
- A ray along the *window* starting at $(1, D)$ with **decreasing** ratio for s beyond $(1, D)$ to infinity ■
- Worst-case ratio is attained for all s between $(1, b)$ and $(1, D)$ ■



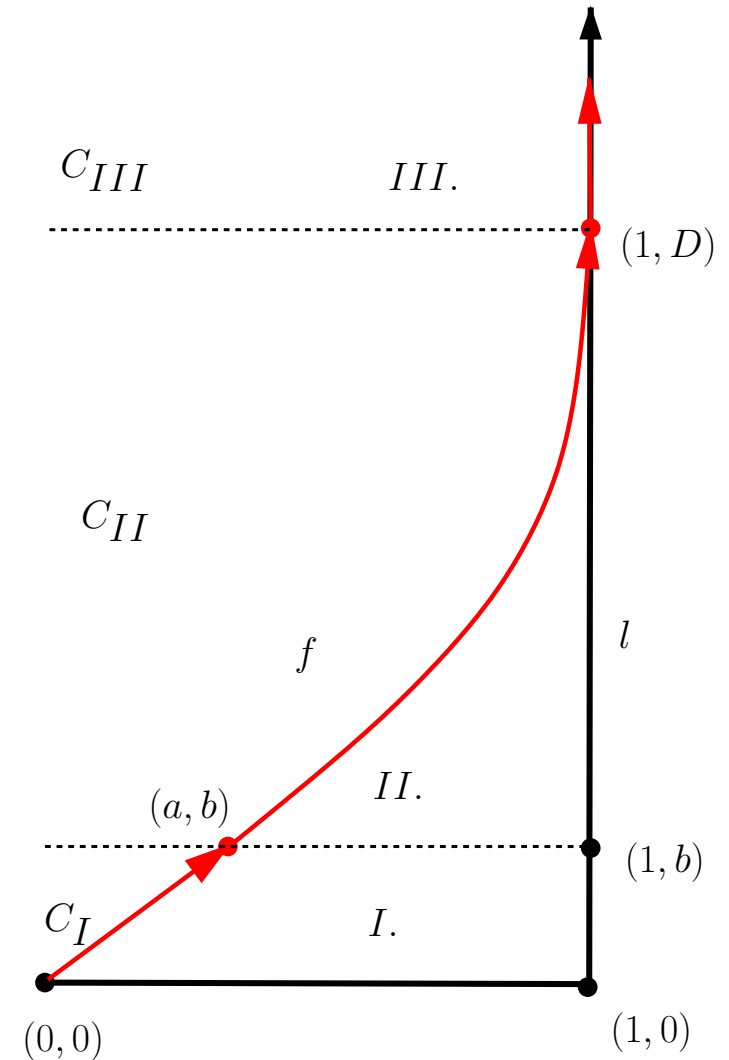
Optimality of this strategy

- By construction
- Curve has the given property
- Proof: Curve is convex
- Assume: Optimal curve K
- K hits ray B at some point (x, b)
- Two cases:
 - Hits B to the left of a : ratio is bigger
 - Cross f beyond B from the right: ratio is bigger



Design of the strategy: By conditions

- 1) Monotonically increasing ratio for s from $(1, 0)$ to $(1, b)$
- 2) Constant ratio for s from $(1, b)$ to $(1, D)$
- Determines a , b and D



Design of the strategy: Condition 1)

- Start with 1): Ratio for $t \in [0, 1]$:■

$$\phi(t) = \frac{t\sqrt{a^2+b^2+1-ta}}{\sqrt{1+t^2b^2}} \quad \blacksquare$$

- Monotonicity: $\phi'(t) \geq 0 \quad \forall t \in [0, 1]$ ■

- Analysis:

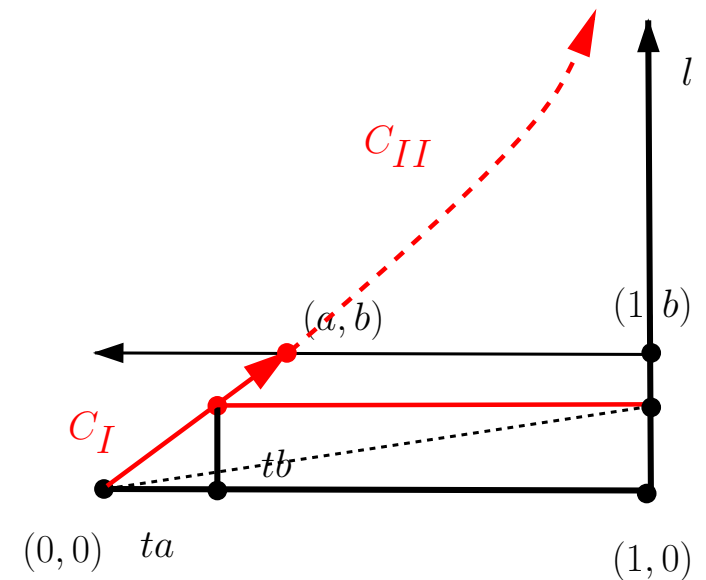
$$\Leftrightarrow \sqrt{a^2 + b^2} - a \geq tb^2 \quad \forall t \in [0, 1] \quad \blacksquare$$

- $\Leftrightarrow b^2 \leq 1 - 2a$ ■

- Choose: $a = \frac{1-b^2}{2}$ ■

- Worst-case ratio:

$$C = \frac{\sqrt{a^2+b^2+1-a}}{\sqrt{1+b^2}} = \sqrt{1+b^2} \quad \blacksquare$$



Design of the strategy: Condition 2)

- 2) Constant ratio $C = \sqrt{1 + b^2}$ for s from $(1, b)$ to $(1, D)$ ■

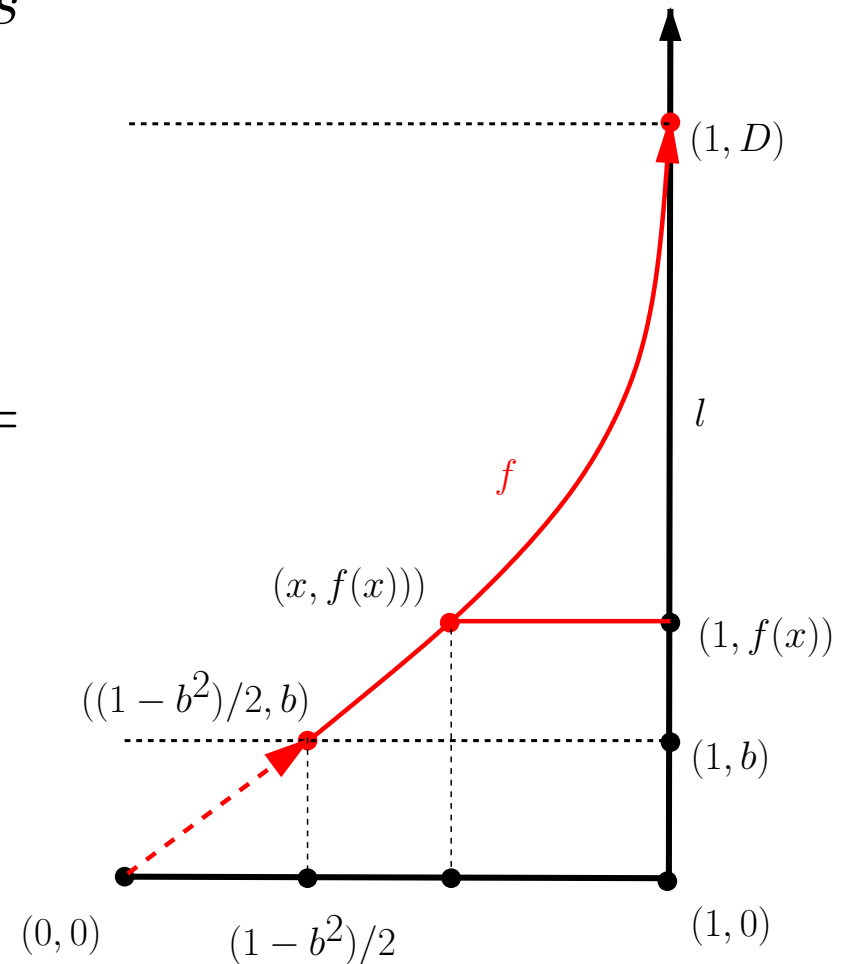
- Function $f(x)$ for $x \in [a, 1]$ ■

- Constant ratio C : ■

$$\sqrt{a^2 + b^2} + \int_a^x \sqrt{1 + f'(t)^2} dt + 1 - x = C \cdot \sqrt{1 + f(x)^2} \blacksquare$$

- Transformations ($f'(x) \neq 0!$): ■

$$\Leftrightarrow f'(x) = 2C \frac{\sqrt{1 + f(x)^2} f(x)}{1 + (1 - C^2) f(x)^2}$$



Solutions for $y = f(x)$

- $f'(x) = 2\sqrt{1+b^2} \frac{\sqrt{1+f(x)^2} f(x)}{1-b^2 f(x)^2}$, $((1-b^2)/2, b)$ on the curve
- Solve: $y' = 1 \cdot 2\sqrt{1+b^2} \frac{\sqrt{1+y^2} y}{1-b^2 y^2}$ for y with $((1-b^2)/2, b)$
- First order diff. eq. $y' = h(x)g(y)$, separated variables, point (k, l)
- Solution: $\int_l^y \frac{dt}{g(t)} = \int_k^x h(z) dz$

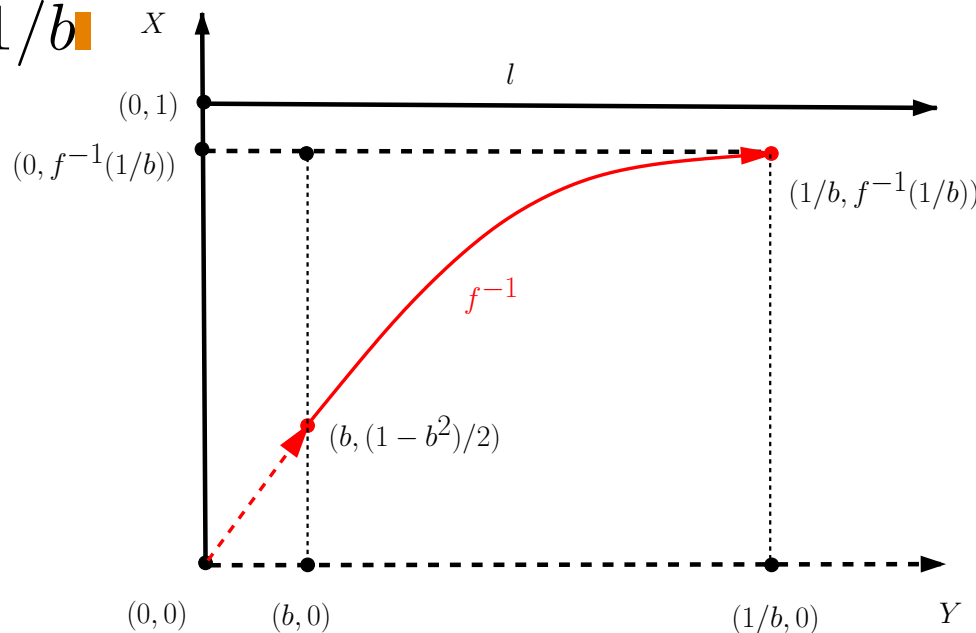
$$\int_b^y \frac{1-b^2 t^2}{2\sqrt{1+b^2}\sqrt{1+t^2} t} dt = \int_{(1-b^2)/2}^x 1 \cdot dz = x - (1-b^2)/2$$

$$x = \frac{b^2 \sqrt{1+y^2} + \operatorname{arctanh}\left(\frac{1}{\sqrt{1+y^2}}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+b^2}}\right) - \sqrt{1+b^2}}{2\sqrt{1+b^2}}$$

- Solution for inverse function $x = f^{-1}(y)$, for $y \in [b, 1/b]$

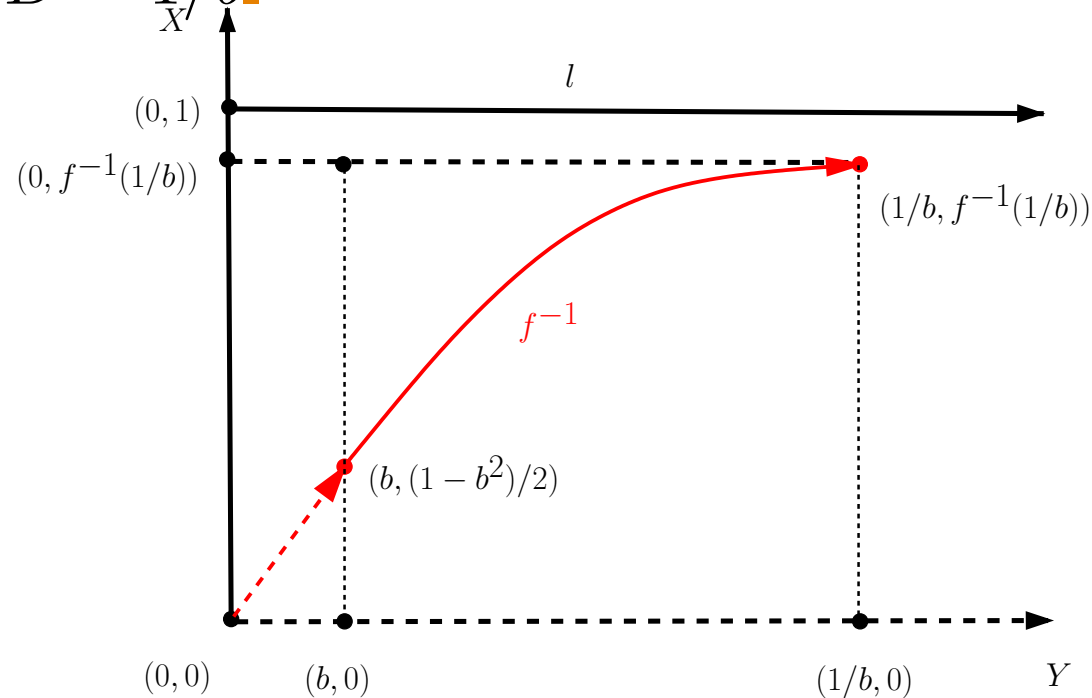
Consider inverse function $x = f^{-1}(y)$

- $x' = \frac{1}{g(y)} = -\frac{(b^2y^2-1)}{2\sqrt{1+y^2}y\sqrt{(1+b^2)}} \geq 0$ for $y \in [b, 1/b]$ ■
- $x'' = -\frac{(b^2y^2+2y^2+1)}{2(1+y^2)^{3/2}\sqrt{1+b^2}y^2} \leq 0$ for $y \geq 0$ ■
- $x = f^{-1}(y)$ concave, $y = f(x)$ convex ■
- Max. at $y = 1/b$ ■



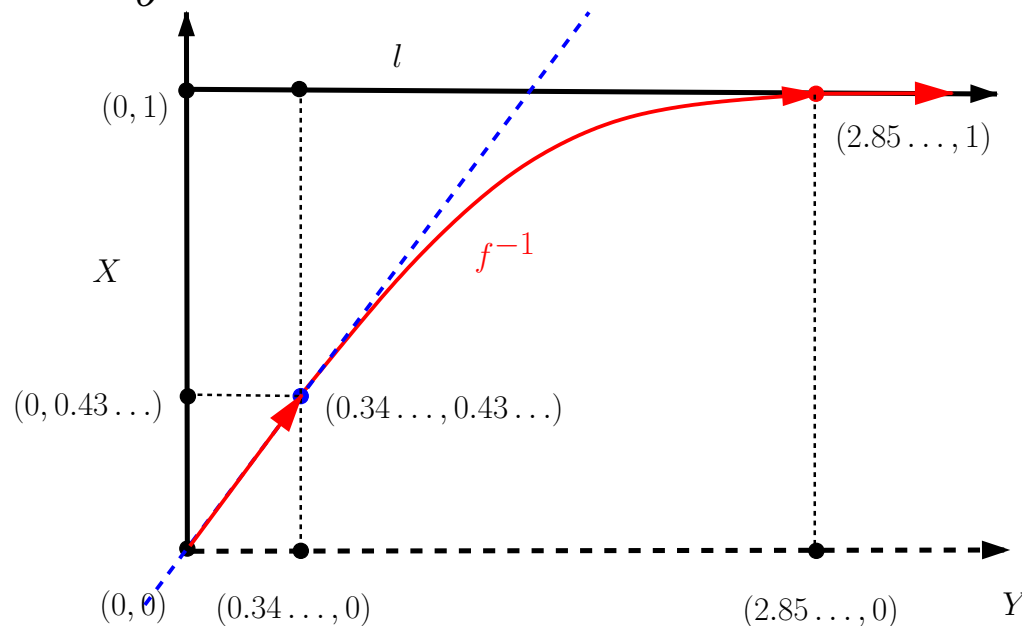
Consider inverse function $x = f^{-1}(y)$

- Maximum at $y = 1/b$ ■
- • Find b so that $f^{-1}(1/b) = 1$ ■
- Fixes b and $D = 1/b$ ■



Optimality of f (or f^{-1})

- Solve $f^{-1}(1/b) = 1$: $b = 0.3497\dots$, $D = 1/b = 2.859\dots$,
- $a = 0.43\dots$, worst-case ratio $C = \sqrt{1 + b^2} = 1.05948\dots$
- f convex from (a, b) to $(1, D)$, line segment convex
- Prolongation of line segment is tangent of f^{-1} at (b, a)
- Insert: $f^{-1}'(b) = \frac{a}{b}$



Conclusion

- Optimal strategy with ratio $C = 1.05948 \dots$

