Online Motion Planning MA-INF 1314
General rays!

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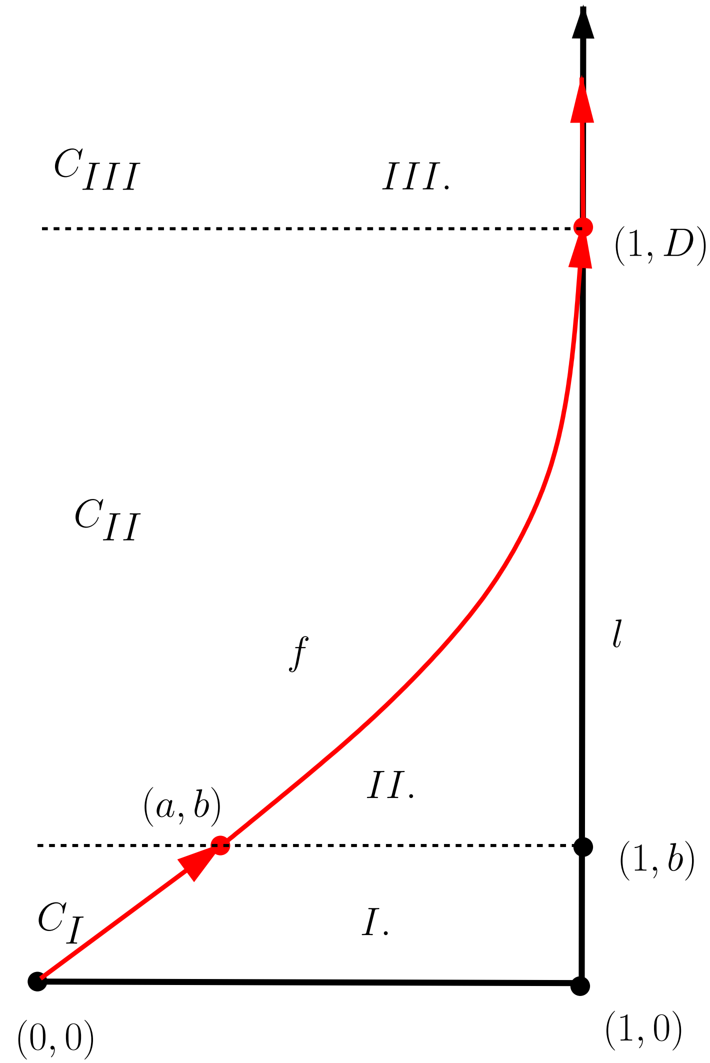
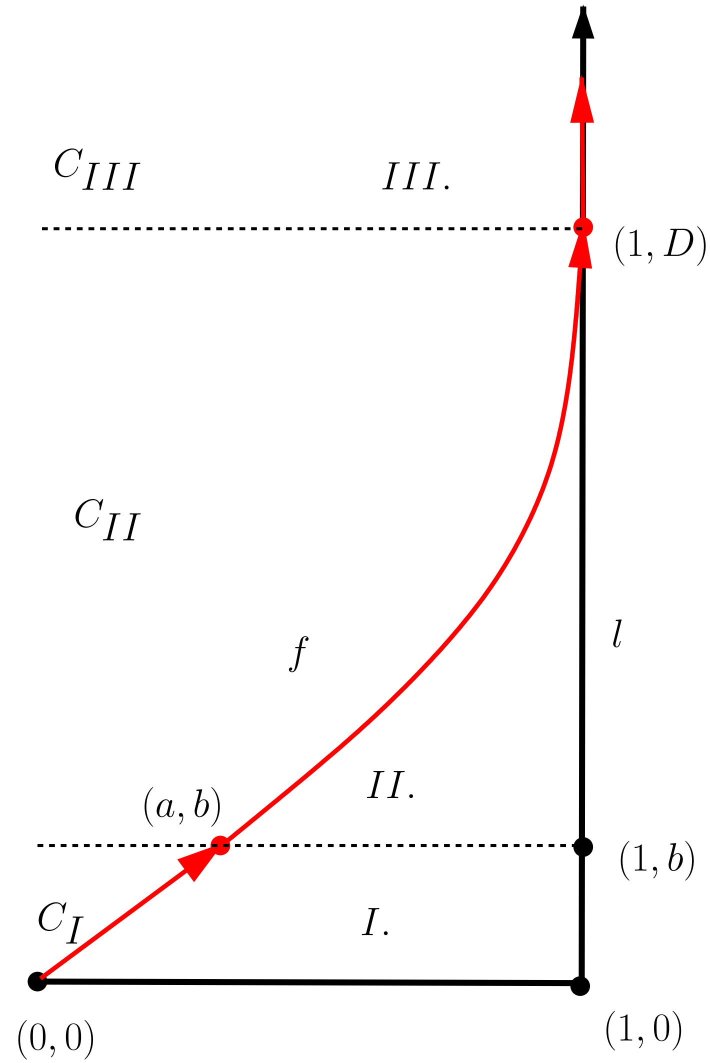
Rep.: Window Shopper Strategy: Three parts

- A line segment from $(0,0)$ to (a,b) with increasing ratio for s between $(1,0)$ and $(1,b)$
- A curve f from (a,b) to some point $(1,D)$ on l which has the same ratio for s between $(1,b)$ and $(1,D)$
- A ray along the *window* starting at $(1,D)$ with decreasing ratio for s beyond $(1,D)$ to infinity
- Worst-case ratio is attained for all s between $(1,b)$ and $(1,D)$
- Optimality by construction, CONVEX!



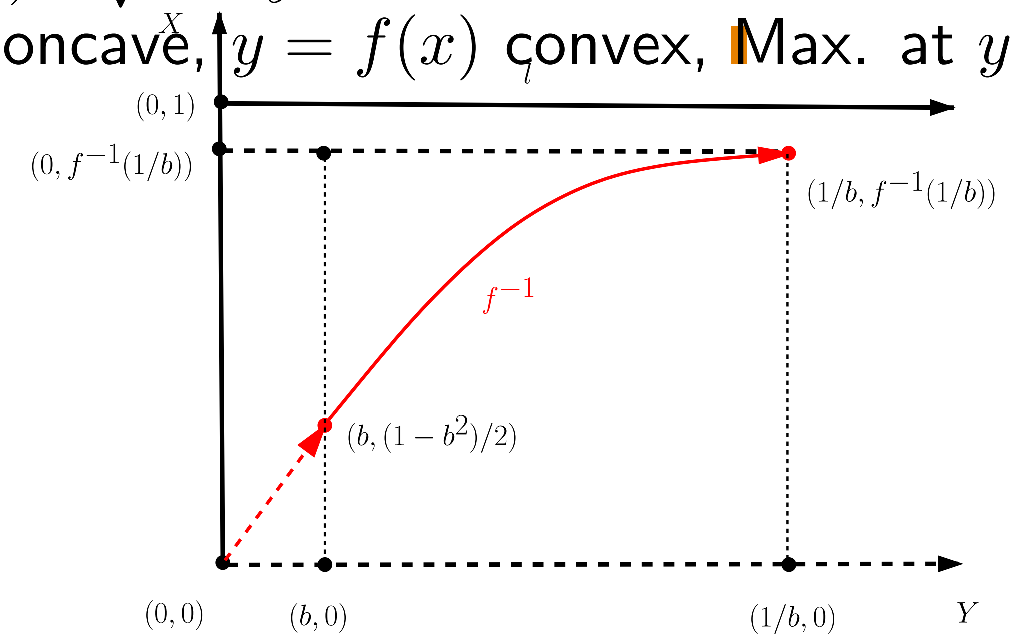
Rep.: Design of the strategy

- Condition I): $a = \frac{1-b^2}{2}$, $\sqrt{1+b^2}$
- Condition II):
  $f'(x) = 2\sqrt{1+b^2}\frac{\sqrt{1+f(x)^2}f(x)}{1-b^2f(x)^2}$,
  Point $((1-b^2)/2, b)$
- Solve: $y' = 1 \cdot 2\sqrt{1+b^2}\frac{\sqrt{1+y^2}y}{1-b^2y^2}$ with $((1-b^2)/2, b)$
- First order diff. eq. $y' = h(x)g(y)$
- Solution: $\int_l^y \frac{dt}{g(t)} = \int_k^x h(z)dz$:
  Here $x = f^{-1}(y)$



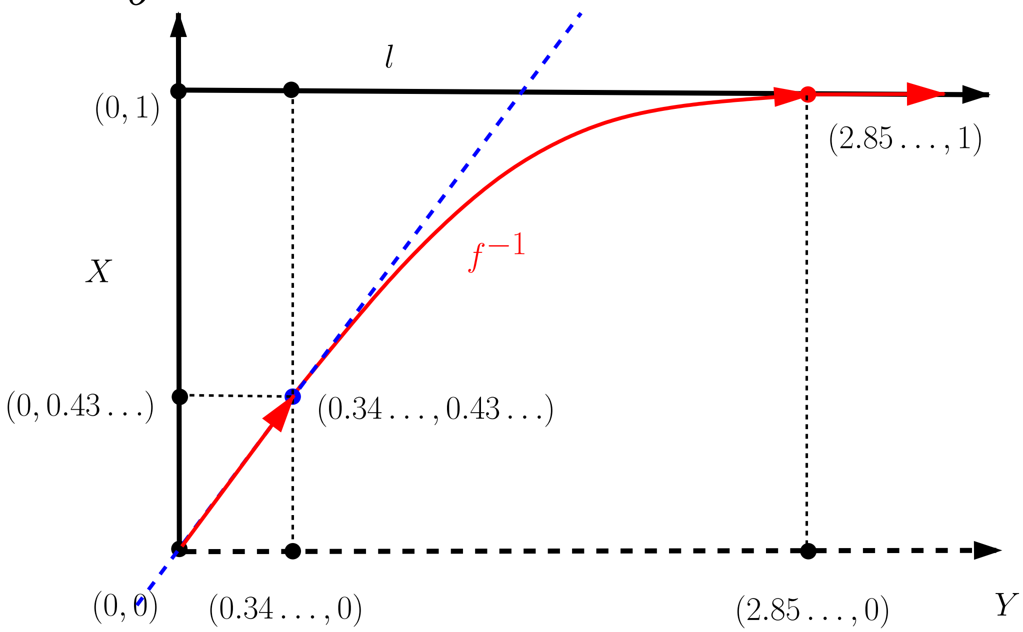
Rep.: Consider inverse function $x = f^{-1}(y)$

- $x = -\frac{b^2\sqrt{1+y^2}+\text{arctanh}\left(1/(\sqrt{1+y^2})\right)-\text{arctanh}\left(1/(\sqrt{1+b^2})\right)-\sqrt{1+b^2}}{2\sqrt{1+b^2}}$
- $x' = \frac{1}{g(y)} = -\frac{(b^2y^2-1)}{2\sqrt{1+y^2}y\sqrt{(1+b^2)}} \geq 0$ for $y \in [b, 1/b]$
- $x'' = -\frac{(b^2y^2+2y^2+1)}{2(1+y^2)^{3/2}\sqrt{1+b^2y^2}} \leq 0$ for $y \geq 0$
- $x = f^{-1}(y)$ concave, $y = f(x)$ convex, Max. at $y = 1/b$



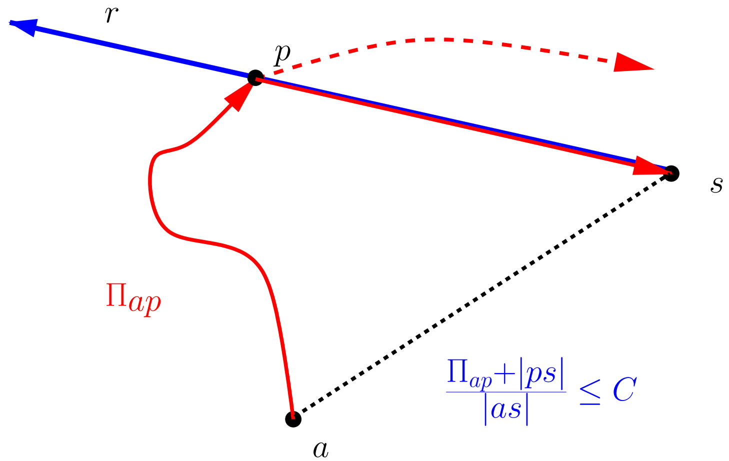
Rep.: Theorem Optimality of f

- Solve $f^{-1}(1/b) = 1$: $b = 0.3497\ldots$, $D = 1/b = 2.859\ldots$, $a = 0.43\ldots$, worst-case ratio $C = \sqrt{1+b^2} = 1.05948\ldots$
- f convex from (a,b) to $(1,D)$, line segment convex
- Prolongation of line segment is tangent of f^{-1} at (b,a)
- Insert: $f^{-1\prime}(b) = \frac{a}{b}$! Convex!



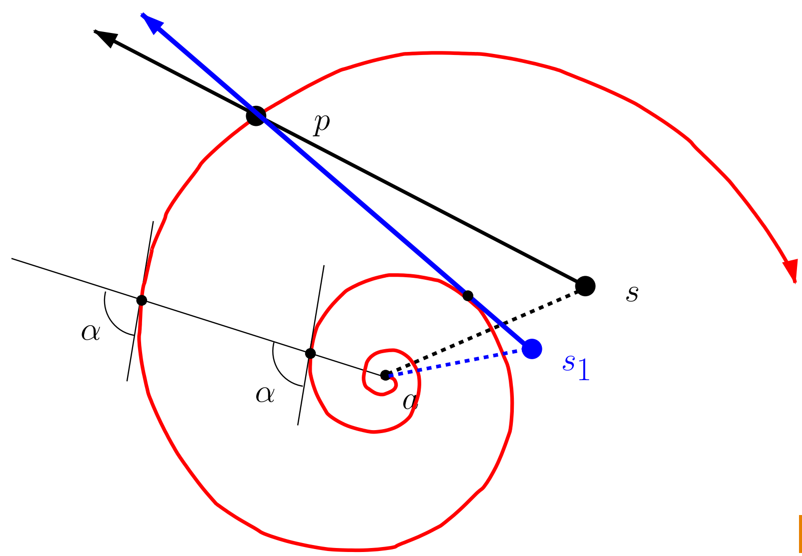
Rays in general

- Rays are somewhere in the plane
- Searchpath Π
- Upper bound: $C = 22.531\ldots$
- Lower bound: $C \geq 2\pi\, e = 17.079\ldots$



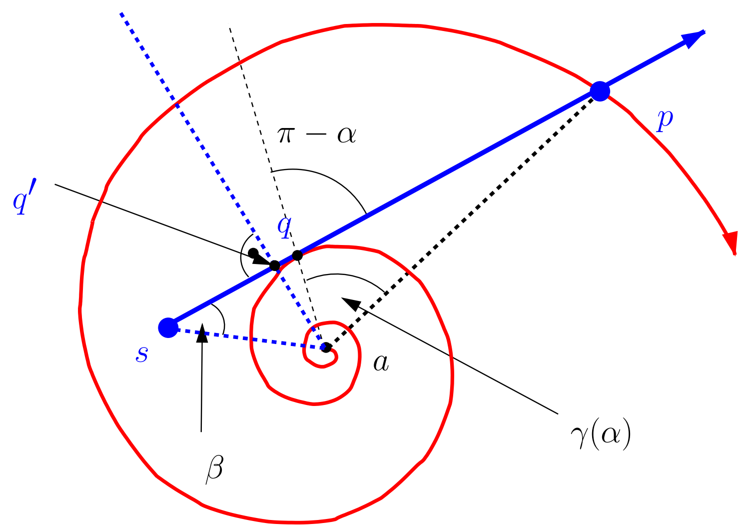
Strategy: Spiral search

- Logarithmic spiral
- Polar coordinates $(\varphi, d \cdot E^{\phi \cot(\alpha)})$, $d > 0, -\infty < \varphi < \infty$
- $\alpha = \pi/2$ Kreis!
- Hits the ray, moves to s
- Worst-case ratio: Ray is a tangent **Lemma 2.38**



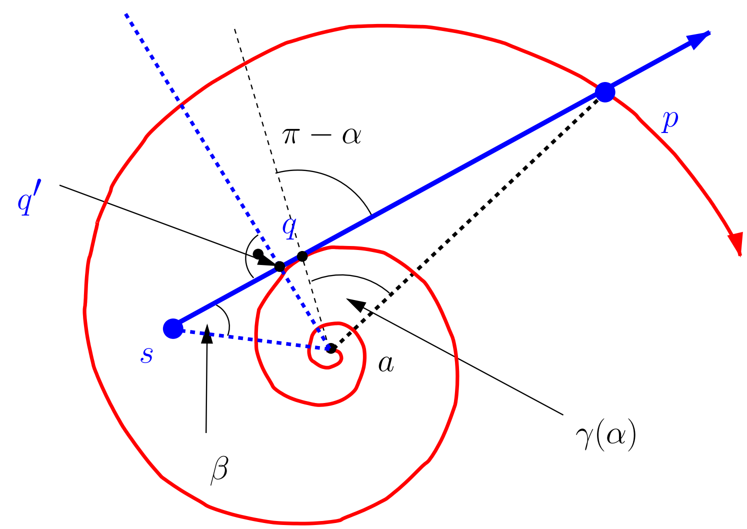
Optimizing the spiral

- Strategy $d \cdot E^{\phi \cot(\alpha)}$, property $|\mathrm{SP}_{\mathtt{a}}^{\mathtt{p}}| = \frac{|\mathtt{ap}|}{|\cos(\alpha)|} = \frac{\mathtt{d}\mathrm{E}^{\theta_{\mathtt{p}} \cot \alpha}}{|\cos(\alpha)|}$
- Ratio C identical for all tangents: Ratio $C(\alpha)$
- We optimize for perpendicular points q'
- Adversary can move s a bit to the left (chooses β)
- Law of sine: Ratio $C(\beta, \alpha) = C(\alpha)\sin(\beta) + \cos(\beta)$



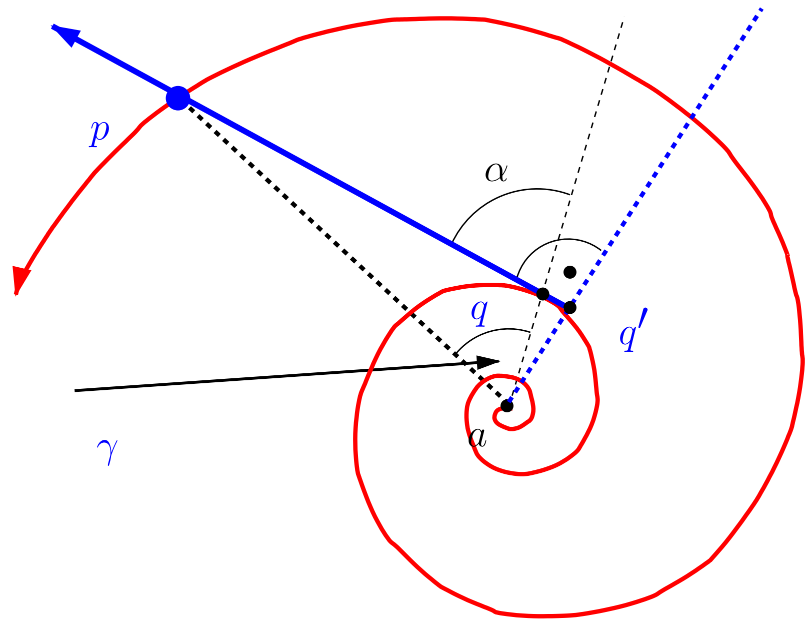
Minimize worst-case ratio

- $dE^{\phi\cot(\alpha))}$: Determine α, $d = 1$
- Assume: Fixed α for q'
- $|as|\sin(\beta) = |aq'|$,
  $|sq'| = |as|\cos(\beta) = \frac{|aq'|\cos(\beta)}{\sin(\beta)}$
- Ratio $C_{q'} = \frac{|\mathtt{SP}_{\mathtt{a}}^{\mathtt{p}}|+|\mathtt{pq}'|}{|aq'|}$
  maximized by
  $\frac{C_{q'} \cdot |aq'|+|aq'|\cos(\beta)/\sin(\beta)}{|aq'|/\sin(\beta)} =$
  $C_{q'}\sin(\beta) + \cos(\beta)$
- Minimize over α for q'
- Finally adverary choose β, fixes s!



Minimize worst-case for q'

- $p = (\phi, E^{\phi \cot(\alpha)})$: Determine $\alpha < \pi/2$
- $|\mathrm{SP}_{\mathtt{a}}^{\mathtt{p}}| = \frac{|\mathtt{ap}|}{\cos(\alpha)} = \frac{\mathrm{E}^{\cot\alpha(2\pi+\gamma+\theta_{\mathtt{q}})}}{\cos(\alpha)}$
- $|pq|\sin(\alpha) = |ap|\sin(\gamma)$ and $|pq| = \frac{E^{\cot\alpha(2\pi+\gamma+\theta_q)}\sin(\gamma)}{\sin\alpha}$
- $|qq'| = |aq|\cos(\alpha)$
- $|pq'| = \frac{E^{\cot\alpha(2\pi+\gamma+\theta_q)}\sin(\gamma)}{\sin\alpha} + E^{\cot\alpha(\theta_q)}\cos(\alpha)$
- $|aq'| = |aq|\sin(\alpha) = E^{\cot\alpha(\theta_q)}\sin(\alpha)$
- γ depends only on α: Now $\gamma(\alpha)$



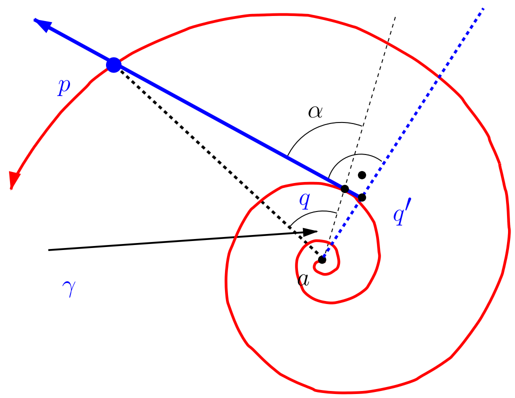
Minimize worst-case for q'

Determine $C_{q'}(\alpha) = \frac{|\Pi_a^p| + |pq'|}{|aq'|}$!

$$\frac{\frac{1}{\cos\alpha}\,\mathrm{E}^{\cot\alpha(\theta_q+2\pi+\gamma(\alpha))} + \frac{1}{\sin\alpha}\,\mathrm{E}^{\cot\alpha(\theta_q+2\pi+\gamma(\alpha))}\sin\gamma(\alpha) + \mathrm{E}^{\cot\alpha\theta_q}\cos\alpha}{\mathrm{E}^{\cot\alpha\theta_q}\sin\alpha} =$$

$$\frac{\frac{1}{\cos\alpha}\,\mathrm{E}^{\cot\alpha(2\pi+\gamma(\alpha))} + \frac{1}{\sin\alpha}\,\mathrm{E}^{\cot\alpha(2\pi+\gamma(\alpha))}\sin\gamma(\alpha) + \cos\alpha}{\sin\alpha} =$$

$$\left(\frac{1}{\sin\alpha\cdot\cos\alpha} + \frac{\sin\gamma(\alpha)}{\sin^2\alpha}\right)\mathrm{E}^{b(2\pi+\gamma(\alpha))} + \cot\alpha$$



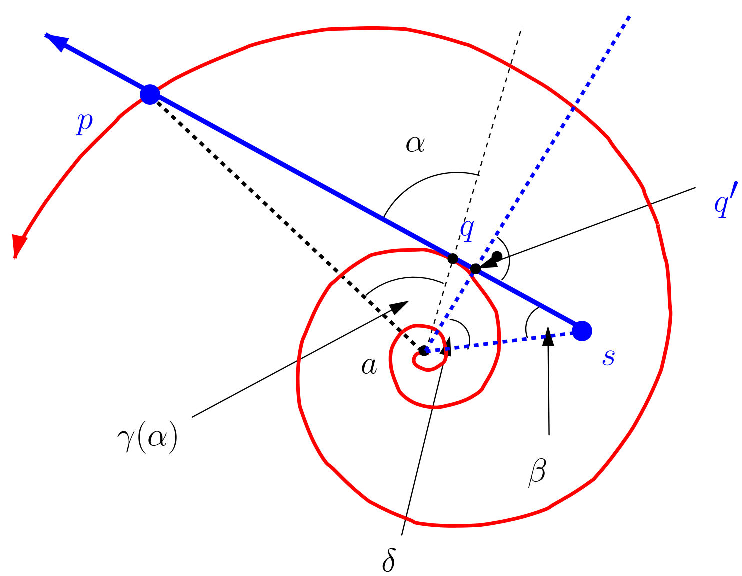
Minimize worst-case for $s = q'$

Determine $\gamma(\alpha)$, Exercise!

Solve Equation: $\frac{\sin\alpha}{\sin(\alpha-\gamma(\alpha))} = E^{\cot\alpha(2\pi+\gamma(\alpha))}$

Then optimize: $f(\alpha) := \left(\frac{1}{\sin\alpha\cdot\cos\alpha} + \frac{\sin\gamma(\alpha)}{\sin^2\alpha}\right) \mathrm{E}^{b(2\pi+\gamma(\alpha))} + \cot\alpha$

Then minimize: $g(\beta) := f(\alpha_{\min})\sin\beta + \cos\beta$!

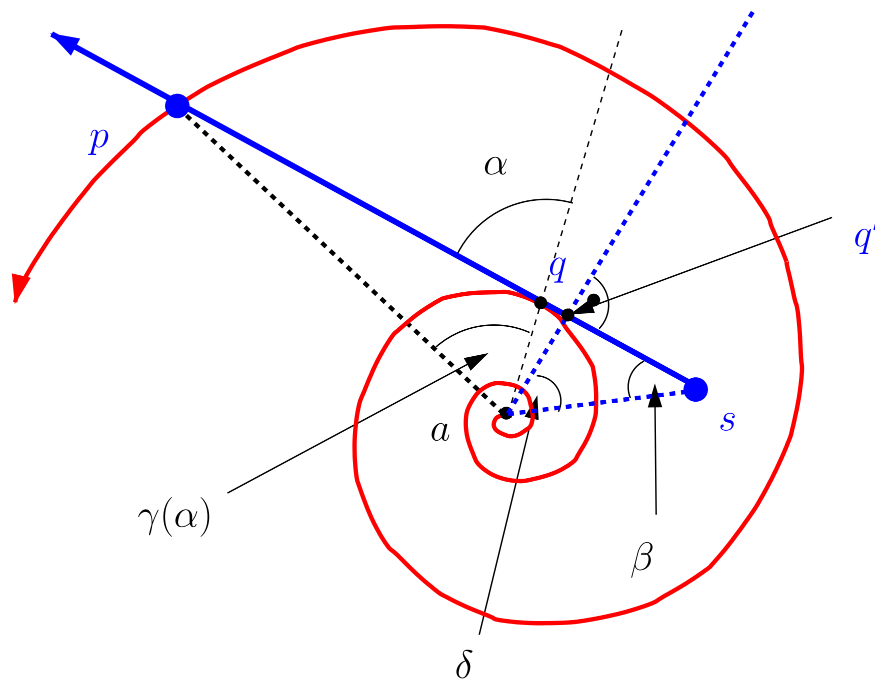


Optimizing the spiral: Theorem 2.42

- Ratio: $C(\alpha)$ for $s = q'$ minimal for $\alpha = 1.4575\ldots$
- $C(\alpha) = 22.4908\ldots$
- Adversary choose β for max. $D(\beta,\alpha) = C(\alpha)\sin(\beta) + \cos(\beta)$
- For $\alpha = 1.4575\ldots$

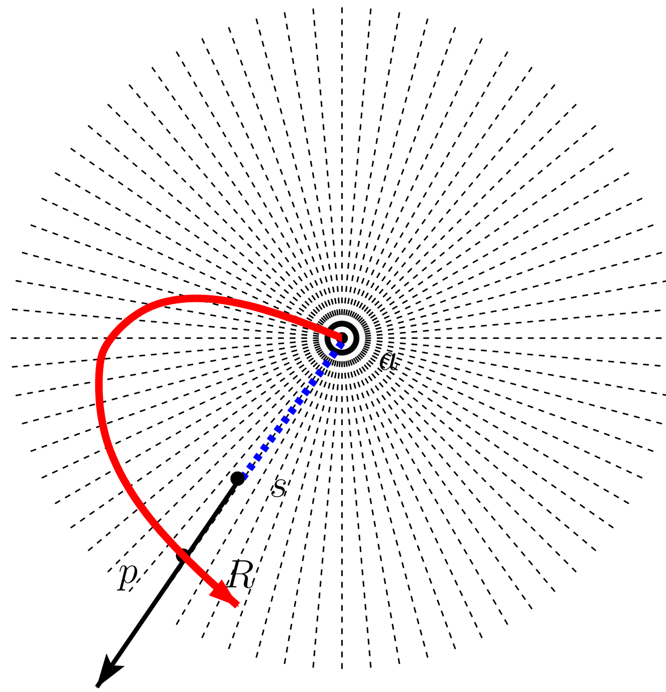
  choose $\delta = 0.044433\ldots$

  $D(\beta, C(\alpha)) = 22.51306056\ldots$



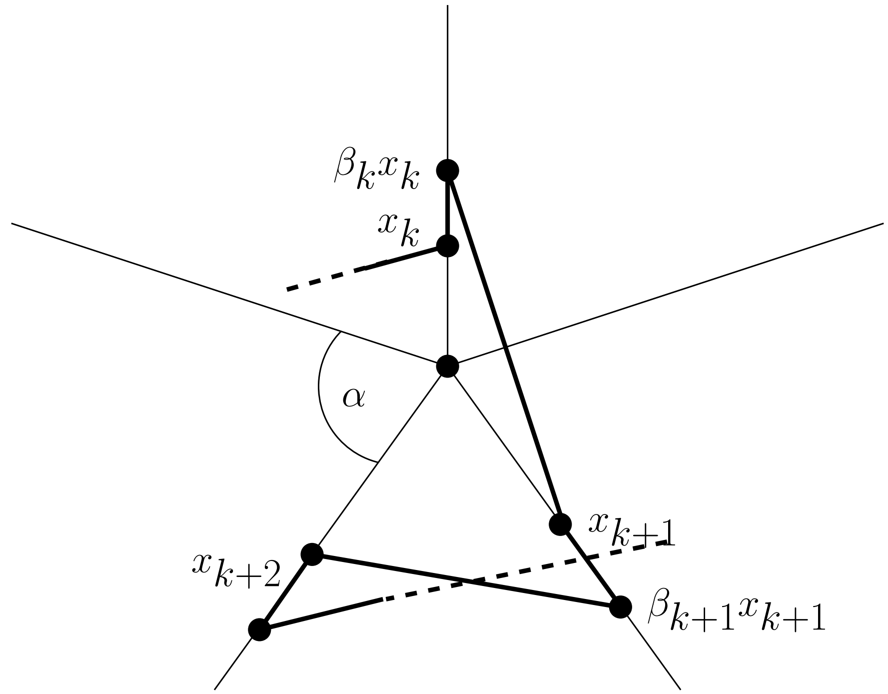
Lower bound: Special example

- General problem of lower bounds
- Special example: Searching for a special ray in the plane
- Cross R and detect s
- Special case: No move to s
- Alpern/Gal (Spiral search): Ratio $C = 17.289\ldots$
- Best ratio among monotone and periodic strategies
- "Complicated task": There is an optimal periodic/monotone strategy



Lower bound construction

- n known rays emanating from a
- Angle $\alpha = \frac{2\pi}{n}$
- Find s on one of the rays
- Also non-periodic and non-monotone strategies
- Stragegy S: Visits the rays in some order
- Hit x_k, leave $\beta_k x_k$ ($\beta_k \geq 1$)

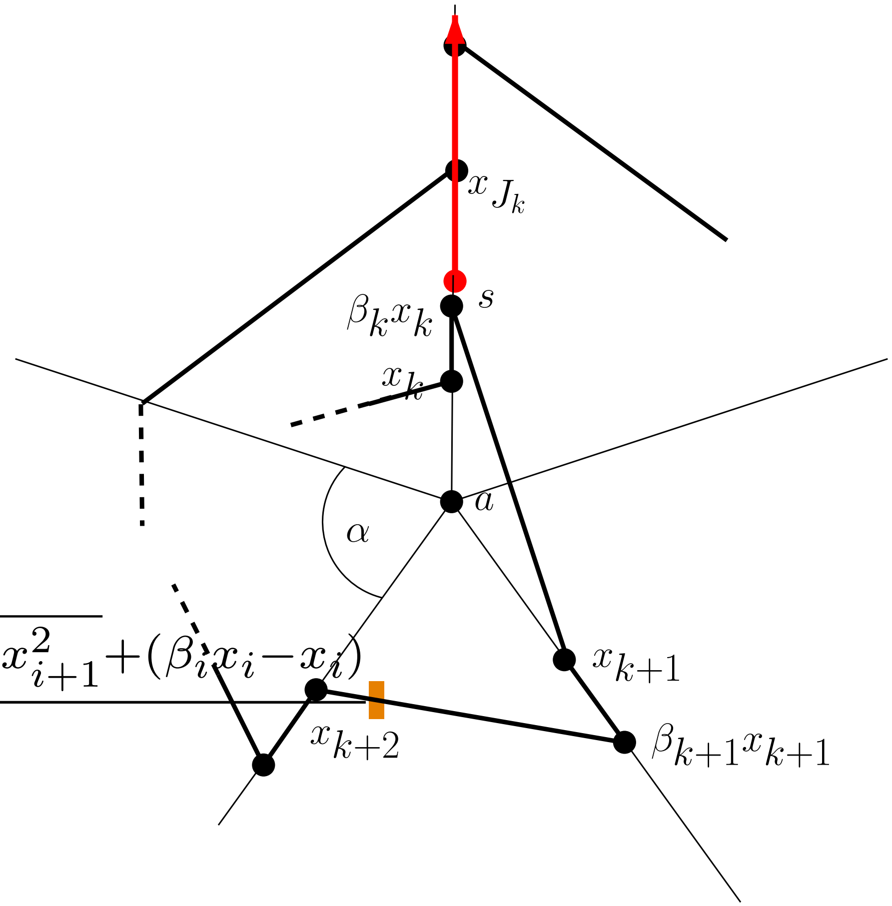


Lower bound construction

- Find s on a ray visited up to $\beta_k x_k$ at the last time, now at x_{J_k}
- Note: Any order is possible
- Worst-case, s close to $\beta_k x_k$
- Ratio: $C(S)$

$$\frac{\sum_{i=1}^{J_k-1}\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1}\cos\gamma_{i,i+1} + x_{i+1}^2} + (\beta_i x_i - x_i)}{\beta_k x_k}$$

- Monotone/Periodic, Funktional??



Lower bound construction

- Ratio: $C(S)$

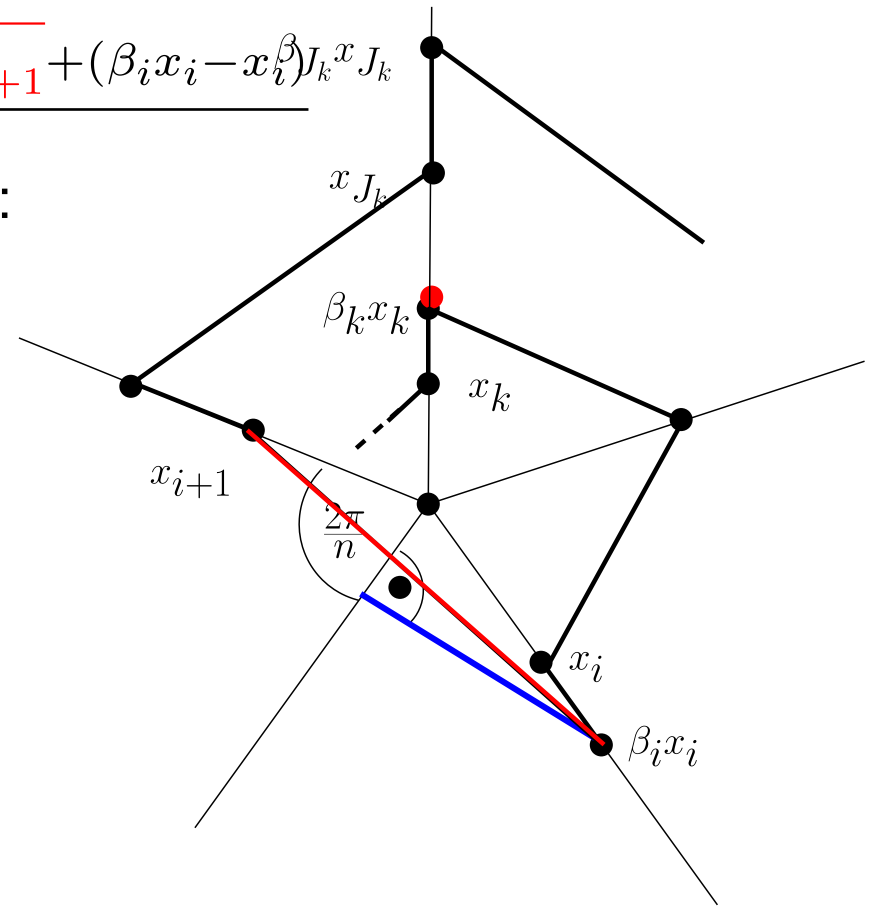
$$\frac{\sum_{i=1}^{J_k-1}\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1}\cos\gamma_{i,i+1} + x_{i+1}^2} + (\beta_i x_i - x_i)}{\beta_k x_k}$$

- Shortest distance to next ray: $\beta_i x_i \sin\frac{2\pi}{n}$
- Lower bound for

$$\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1}\cos\gamma_{i,i+1} + x_{i+1}^2}$$

- Lower bound: $C(S) \geq$

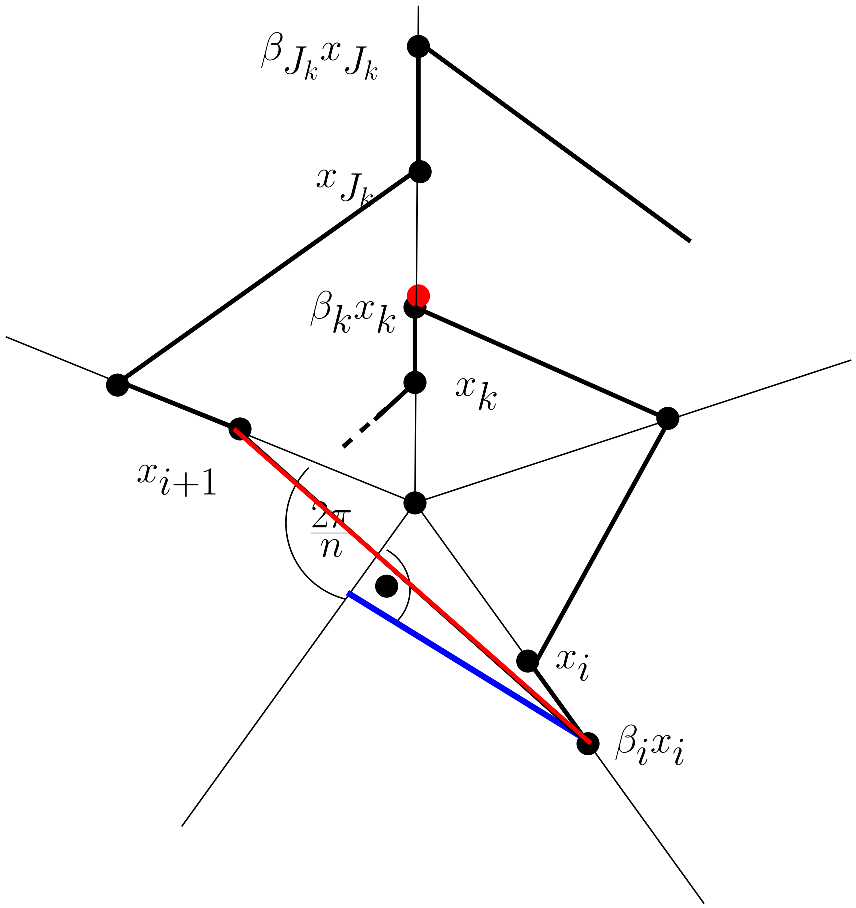
$$\sin\frac{2\pi}{n}\frac{\sum_{i=1}^{J_k-1}\beta_i x_i}{\beta_k x_k}$$



Lower bound construction

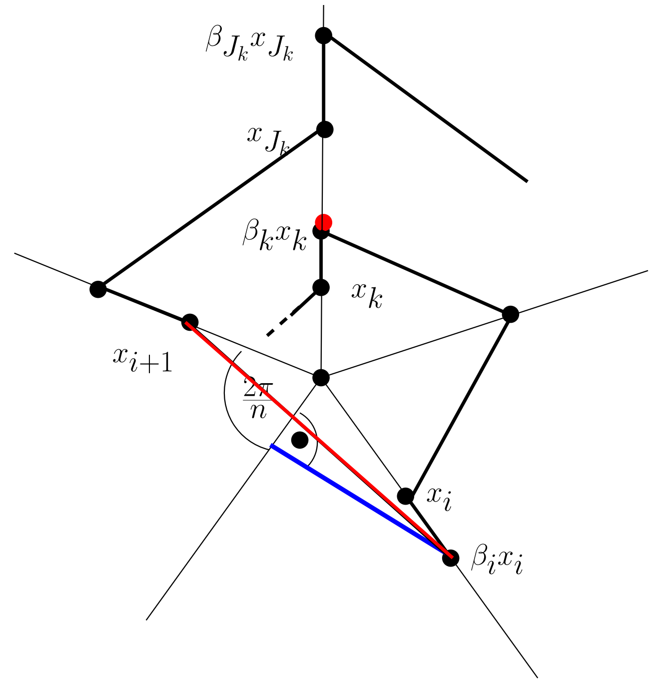
- Lower bound $\frac{\sum_{i=1}^{J_k-1} f_i}{f_k}$
- Equals functional of standard m-ray search
- Optimal strategy: monotone/periodic (Alpern/Gal)
- $f_i = \left(\frac{n}{n-1}\right)^i$,

  ratio: $(n-1)\left(\frac{n}{n-1}\right)^n$
- $C(S) \geq \sin\frac{2\pi}{n}(n-1)\left(\frac{n}{n-1}\right)^n$



Lower bound construction

- $C(S) \geq \sin\frac{2\pi}{n}(n-1)\left(\frac{n}{n-1}\right)^n$



$$\lim_{n\to\infty}(n-1)\left(\frac{n}{n-1}\right)^n \sin\frac{2\pi}{n} =$$

$$2\pi\, e = 17.079\ldots$$

- Lower bound: **Theorem**

Summary

- The Window-Shopper-Problem
- Optimal strategy $C = 1.059\ldots$: **Theorem**
- Interesting design technique
- Rays in general
- Lower $C \geq 2\pi\, e = 17.079\ldots$ (**Theorem**) and upper bound $C = 22.51\ldots$ (**Theorem**)
- Lower bound construction
- Also a lower bound for special case with $C = 17.289\ldots$