

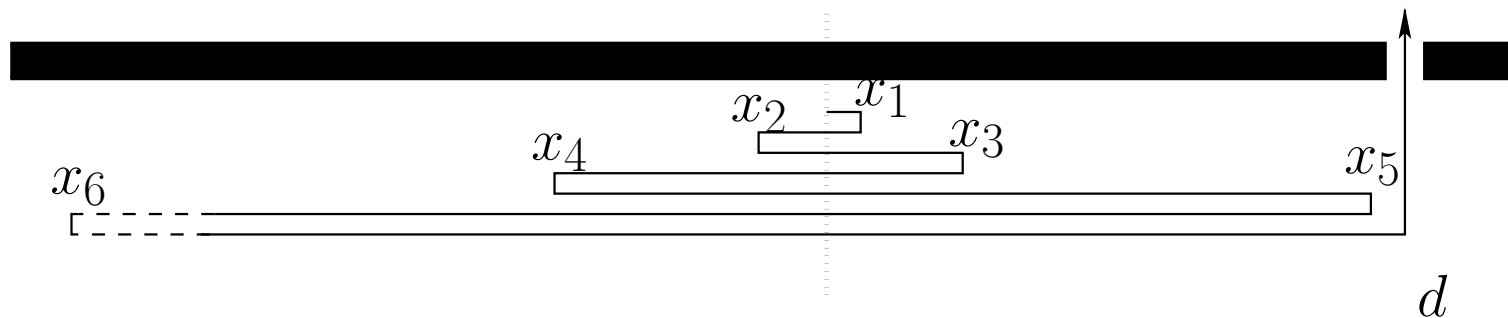
Online Motion Planning MA-INF 1314

Window Shopper

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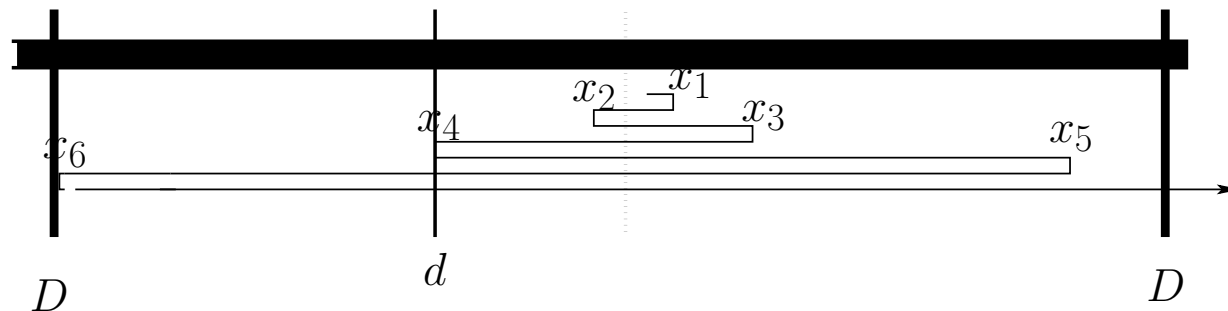
Rep. 2-ray search: Optimality for equations!

- Set: $\frac{\sum_{i=1}^{k+1} x'_i}{x'_k} = \frac{(C-1)}{2}$ for all k
- $\sum_{i=1}^{k+1} x'_i - \sum_{i=1}^k x'_i = \frac{(C-1)}{2} (x'_k - x'_{k-1})$
- Thus: $C' (x'_k - x'_{k-1}) = x'_{k+1}$, Recurrence!
- Solve a recurrence! Analytically! Blackboard!
- Characteristical polynom: No solution $C' < 4$
- $x'_i = (i + 1)2^i$ with $C' = 4$ is a solution! Blackboard! Optimal!



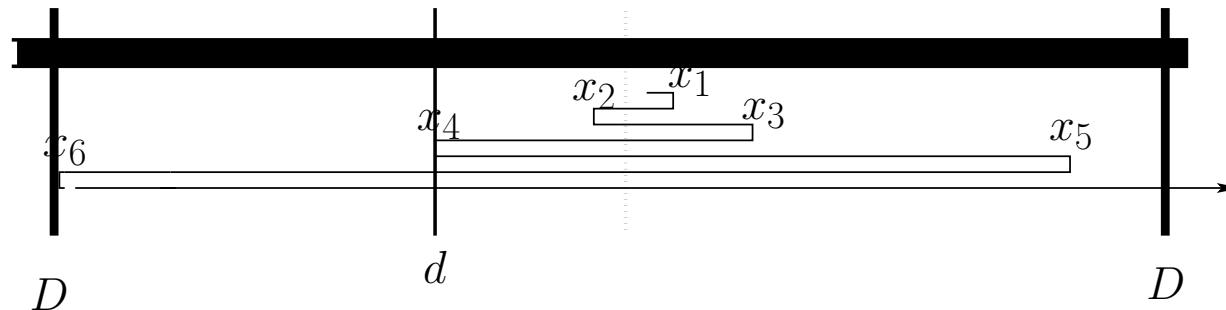
Rep.: 2-ray search, restricted distance

- Assume goal is no more than dist. $\leq D$ away
- Exactly D ! Simple ratio 3!
- Find optimal strategy, minimize C !
- Vice-versa: C is given! Find the largest distance D (reach R) that still allows C competitive search.
- One side with $f_{\text{Ende}} = R$, the other side arbitrarily large!



Rep.: 2-ray search, maximal reach R

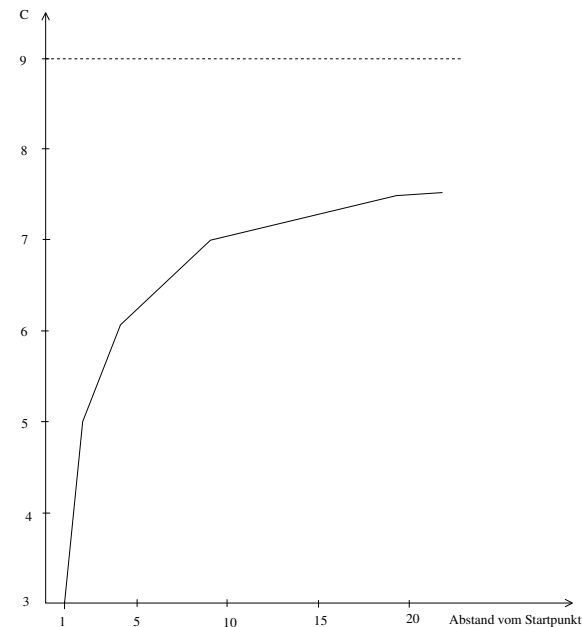
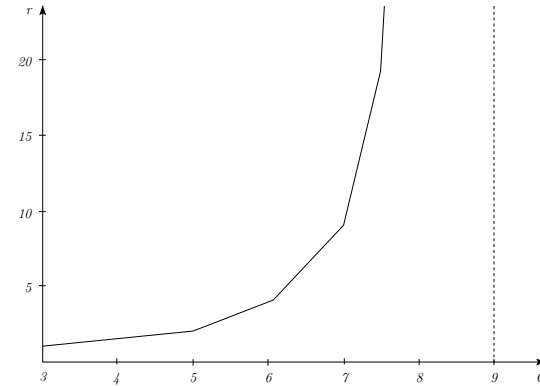
- C given, optimal reach R !
- **Theorem** The strategy with equality in any step maximizes the reach R !
- Strategy: $\frac{\sum_{i=1}^{k+1} x_i}{x_k} = \frac{(C-1)}{2}$, first step: $x_1 = \frac{(C-1)}{2}$
- Recurrence: $x_0 = 1$, $x_{-1} = 0$, $x_{k+1} = \frac{(C-1)}{2}(x_k - x_{k-1})$
- Strategy is optimal! By means of the Comp. Geom. lecture!



Rep: Solutions!

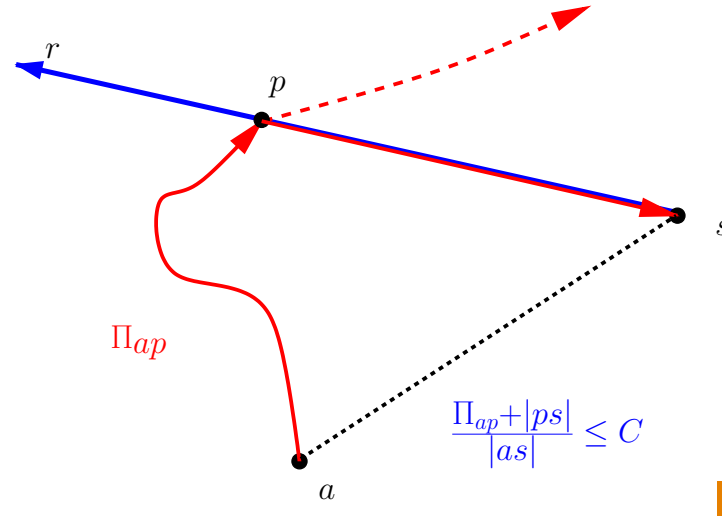


- $f(C) := \text{max. reach}$ depending on C
- Vice versa, R given, binary search



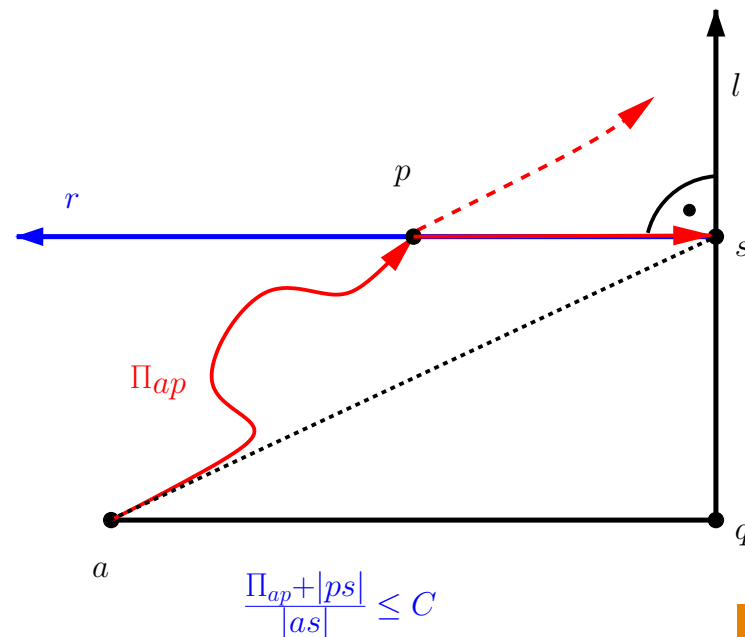
Rep: Searching for the origin of ray

- Unknown ray r in the plane, unknown origin s Startpoint a
- Searchpath Π , hits r , detects s , move to s
- Shortest path OPT, build the ratio
- Π has *competitive ratio* C if inequality holds for all rays
- Task: Find searchpath Π with the minimal C
- Special Problem: Window Shopper



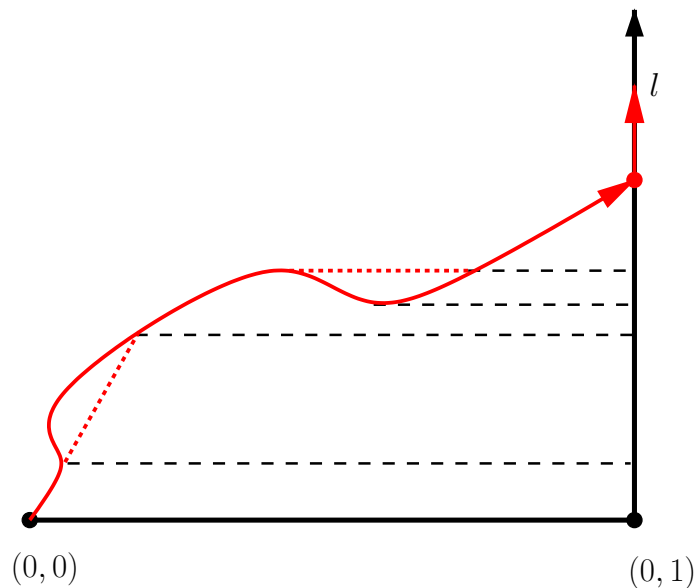
Rep: The Window-Shopper-Problem

- Unknown ray starts at s on *known* vertical line l (window)■
- Ray starts perpendicular to l ■
- aq runs parallel to r ■
- *Motivation:* Move along a window until you *detect* an item■
- Move to the item■



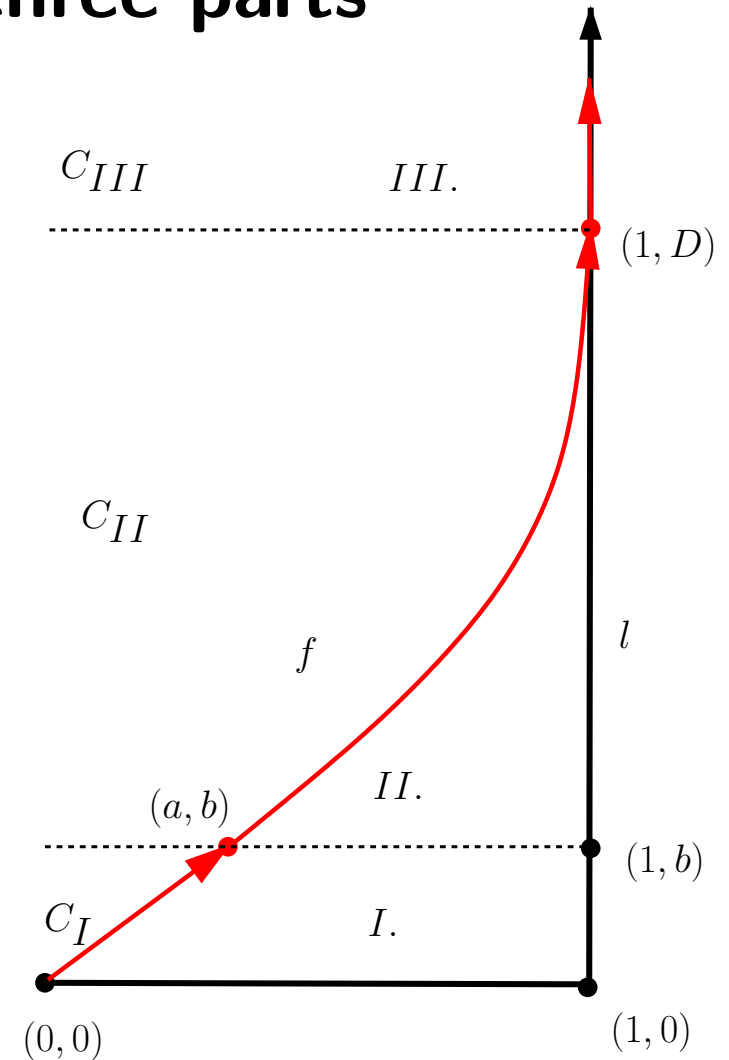
Rep: Some observations

- Any reasonable strategy is monotone in x and y
- Otherwise: Optimize for some s on l
- Finally hits the *window*
- Ratio is close to 1 in the beginning, but bigger than 1
- Ratio goes to 1 at the end

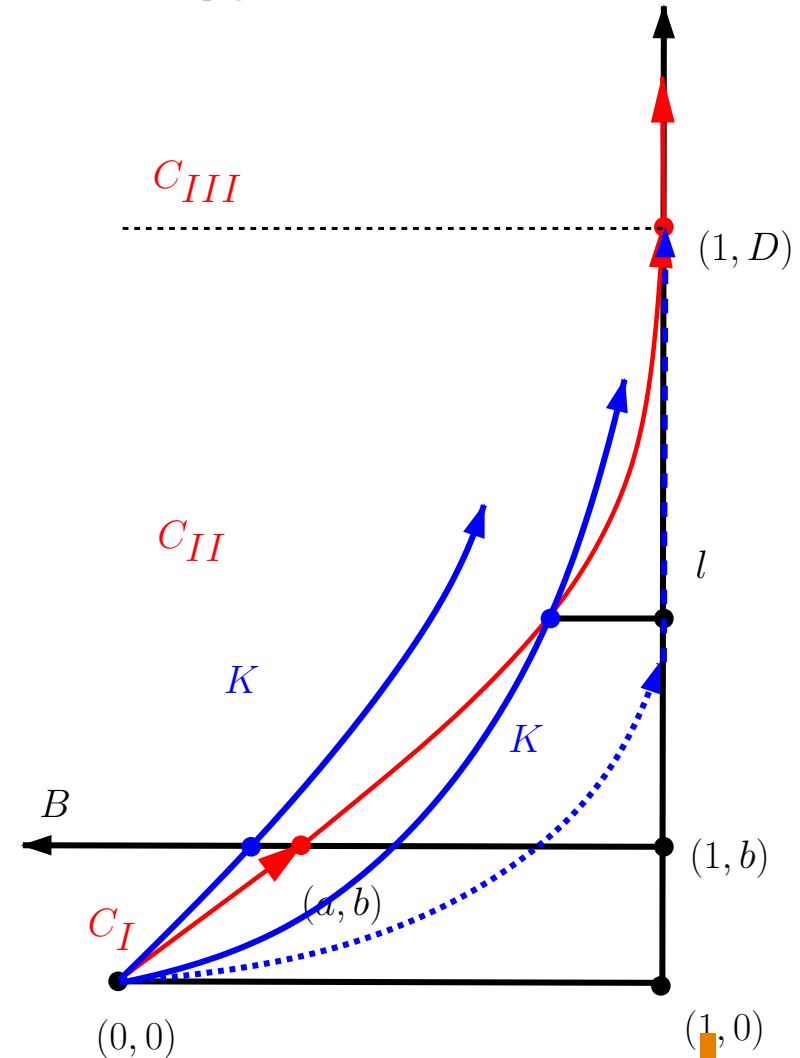


Rep: Strategy design, three parts

- A line segment from $(0, 0)$ to (a, b) with **increasing** ratio for s between $(1, 0)$ and $(1, b)$
- A curve f from (a, b) to some point $(1, D)$ on l which has **the same** ratio for s between $(1, b)$ and $(1, D)$
- A ray along the *window* starting at $(1, D)$ with **decreasing** ratio for s beyond $(1, D)$ to infinity
- Worst-case ratio is attained for all s between $(1, b)$ and $(1, D)$

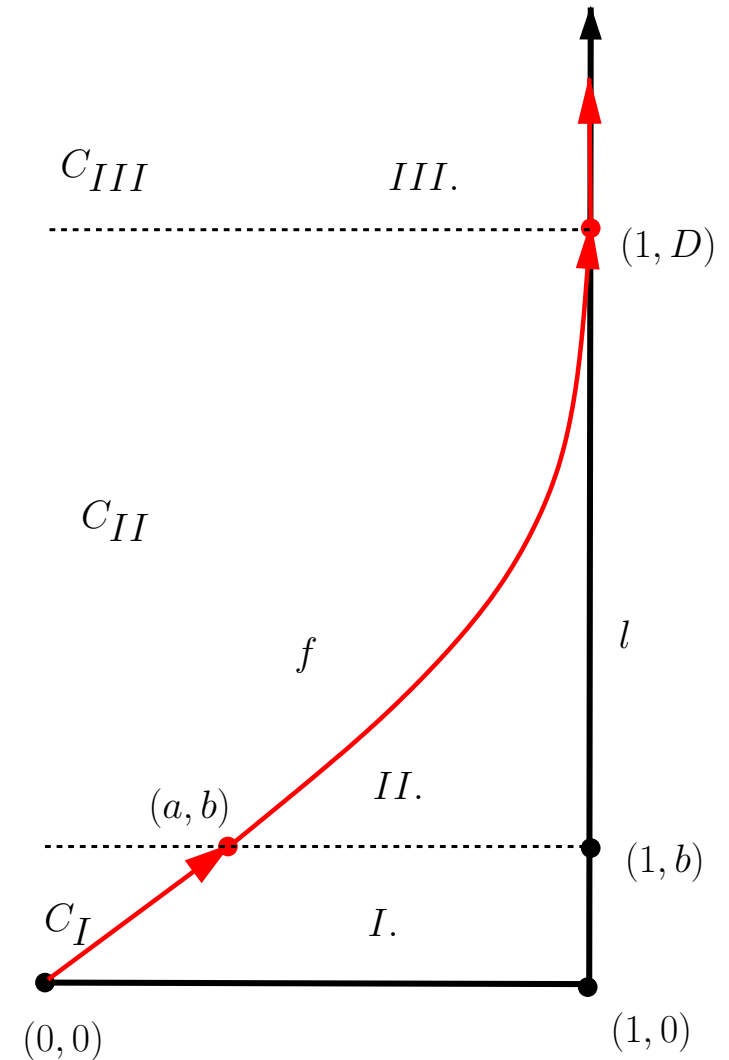


- By construction
- Curve has the given property
- Proof: Curve is convex
- Assume: Optimal curve K
- K hits ray B at some point (x, b)
- Two cases:
 - Hits B to the left of a : ratio is big
 - Cross f beyond B from the right: ratio is bigger



Design of the strategy: By conditions

- 1) Monotonically increasing ratio for s from $(1, 0)$ to $(1, b)$
- 2) Constant ratio for s from $(1, b)$ to $(1, D)$
- Determines a , b and D



Design of the strategy: Condition 1)

- Start with 1): Ratio for $t \in [0, 1]$:

$$\phi(t) = \frac{t\sqrt{a^2+b^2+1-ta}}{\sqrt{1+t^2b^2}}$$

- Monotonicity: $\phi'(t) \geq 0 \quad \forall t \in [0, 1]$

- Analysis:

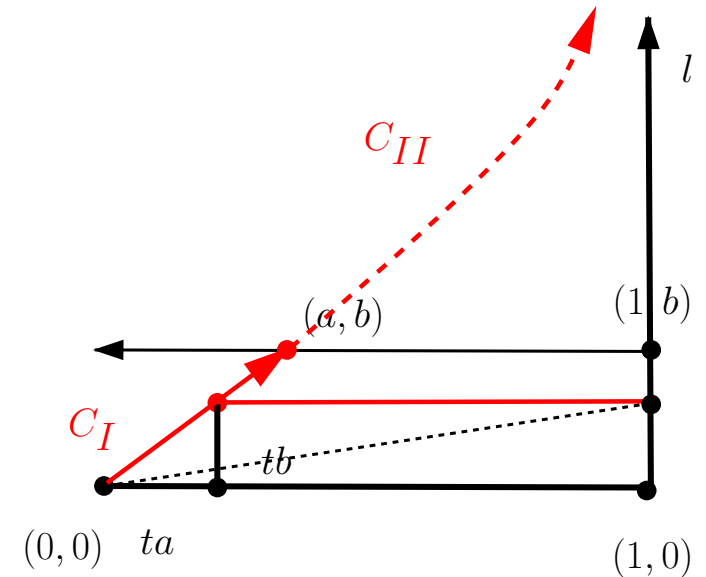
$$\Leftrightarrow \sqrt{a^2 + b^2} - a \geq tb^2 \quad \forall t \in [0, 1]$$

- $\Leftrightarrow b^2 \leq 1 - 2a$

- Choose: $a = \frac{1-b^2}{2}$

- Worst-case ratio:

$$C = \frac{\sqrt{a^2+b^2+1-a}}{\sqrt{1+b^2}} = \sqrt{1+b^2}$$



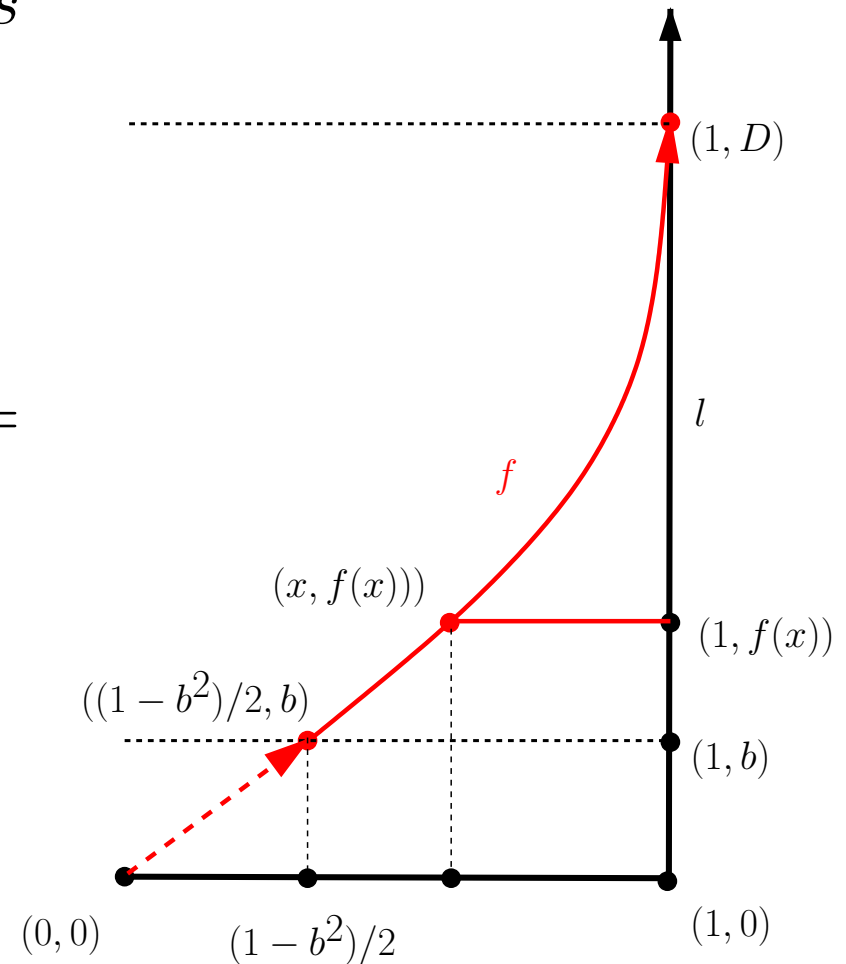
Design of the strategy: Condition 2)

- 2) Constant ratio $C = \sqrt{1 + b^2}$ for s from $(1, b)$ to $(1, D)$ ■
- • Function $f(x)$ for $x \in [a, 1]$ ■
- Constant ratio C : ■

$$\sqrt{a^2 + b^2} + \int_a^x \sqrt{1 + f'(t)^2} dt + 1 - x = C \cdot \sqrt{1 + f(x)^2} \quad \blacksquare$$

- Transformations ($f'(x) \neq 0!$): ■

$$\Leftrightarrow f'(x) = 2C \frac{\sqrt{1 + f(x)^2} f(x)}{1 + (1 - C^2) f(x)^2}$$



Solutions for $y = f(x)$

- $f'(x) = 2\sqrt{1+b^2} \frac{\sqrt{1+f(x)^2} f(x)}{1-b^2 f(x)^2}$, $((1-b^2)/2, b)$ on the curve
- Solve: $y' = 1 \cdot 2\sqrt{1+b^2} \frac{\sqrt{1+y^2} y}{1-b^2 y^2}$ for y with $((1-b^2)/2, b)$
- First order diff. eq. $y' = h(x)g(y)$, separated variables, point (k, l)
- Solution: $\int_l^y \frac{dt}{g(t)} = \int_k^x h(z) dz$

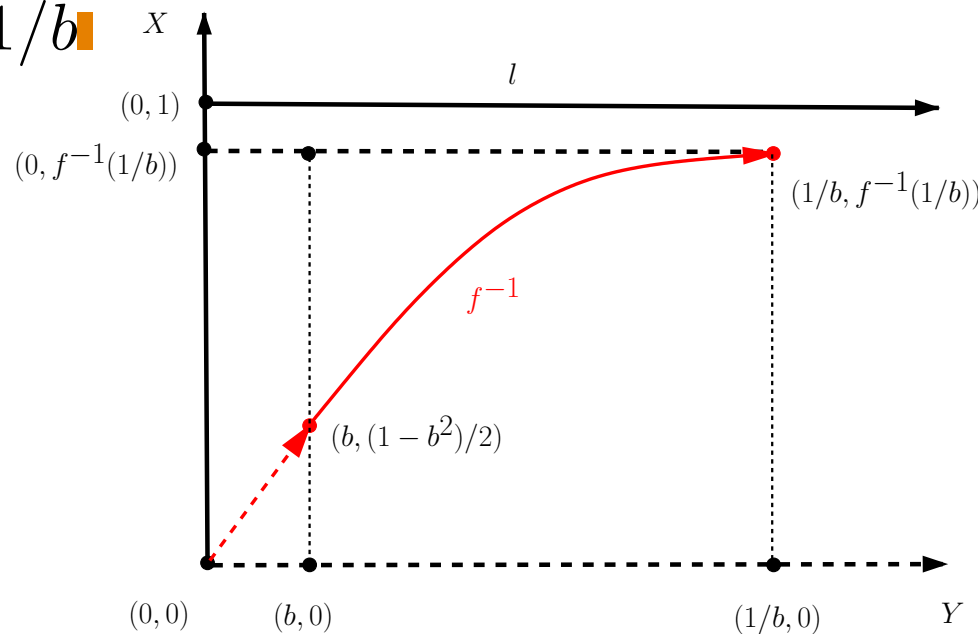
$$\int_b^y \frac{1-b^2 t^2}{2\sqrt{1+b^2}\sqrt{1+t^2}t} dt = \int_{(1-b^2)/2}^x 1 \cdot dz = x - (1-b^2)/2$$

$$x = -\frac{b^2\sqrt{1+y^2} + \operatorname{arctanh}\left(1/(\sqrt{1+y^2})\right) - \operatorname{arctanh}\left(1/(\sqrt{1+b^2})\right) - \sqrt{1+b^2}}{2\sqrt{1+b^2}}$$

- Solution for inverse function $x = f^{-1}(y)$, for $y \in [b, 1/b]$

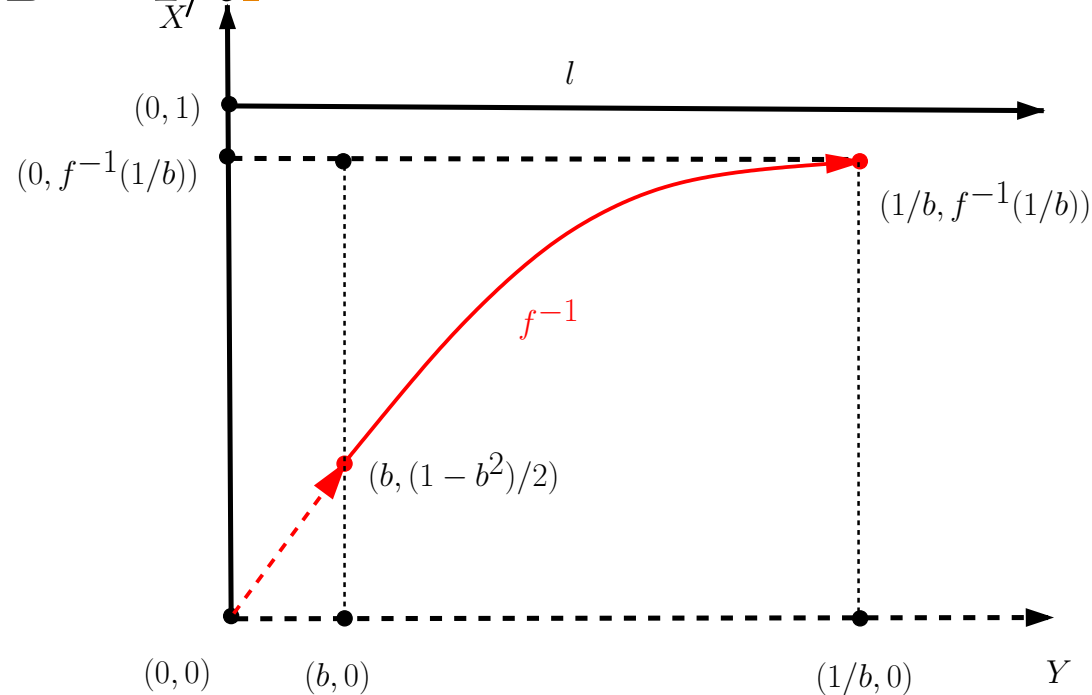
Consider inverse function $x = f^{-1}(y)$

- $x' = \frac{1}{g(y)} = -\frac{(b^2 y^2 - 1)}{2\sqrt{1+y^2}y\sqrt{(1+b^2)}} \geq 0$ for $y \in [b, 1/b]$
- $x'' = -\frac{(b^2 y^2 + 2y^2 + 1)}{2(1+y^2)^{3/2}\sqrt{1+b^2}y^2} \leq 0$ for $y \geq 0$
- $x = f^{-1}(y)$ concave, $y = f(x)$ convex
- Max. at $y = 1/b$



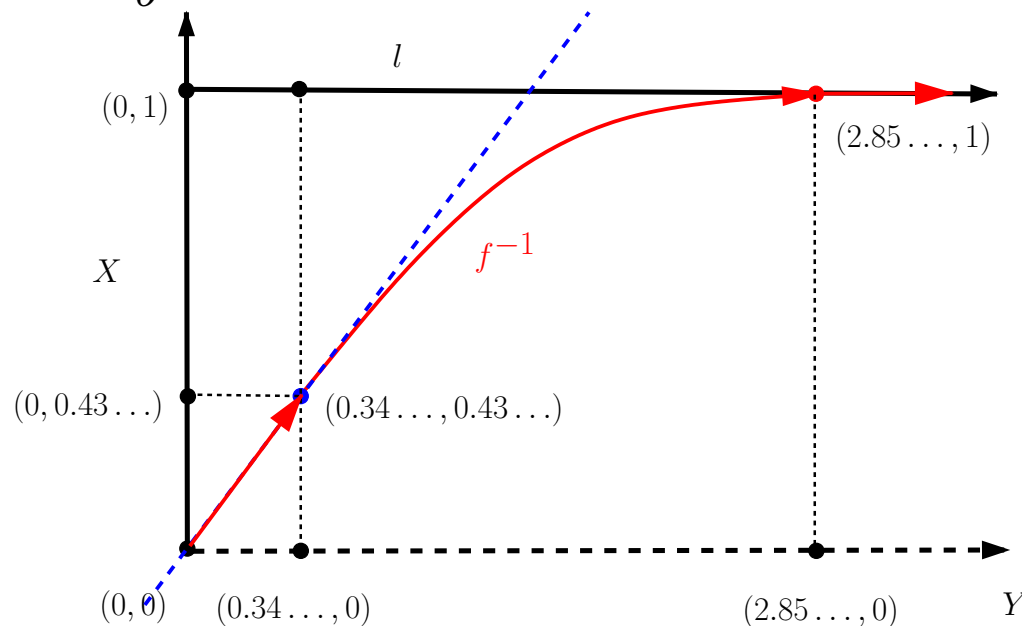
Consider inverse function $x = f^{-1}(y)$

- Maximum at $y = 1/b$
- Find b so that $f^{-1}(1/b) = 1$
- Fixes b and $D = 1/b$



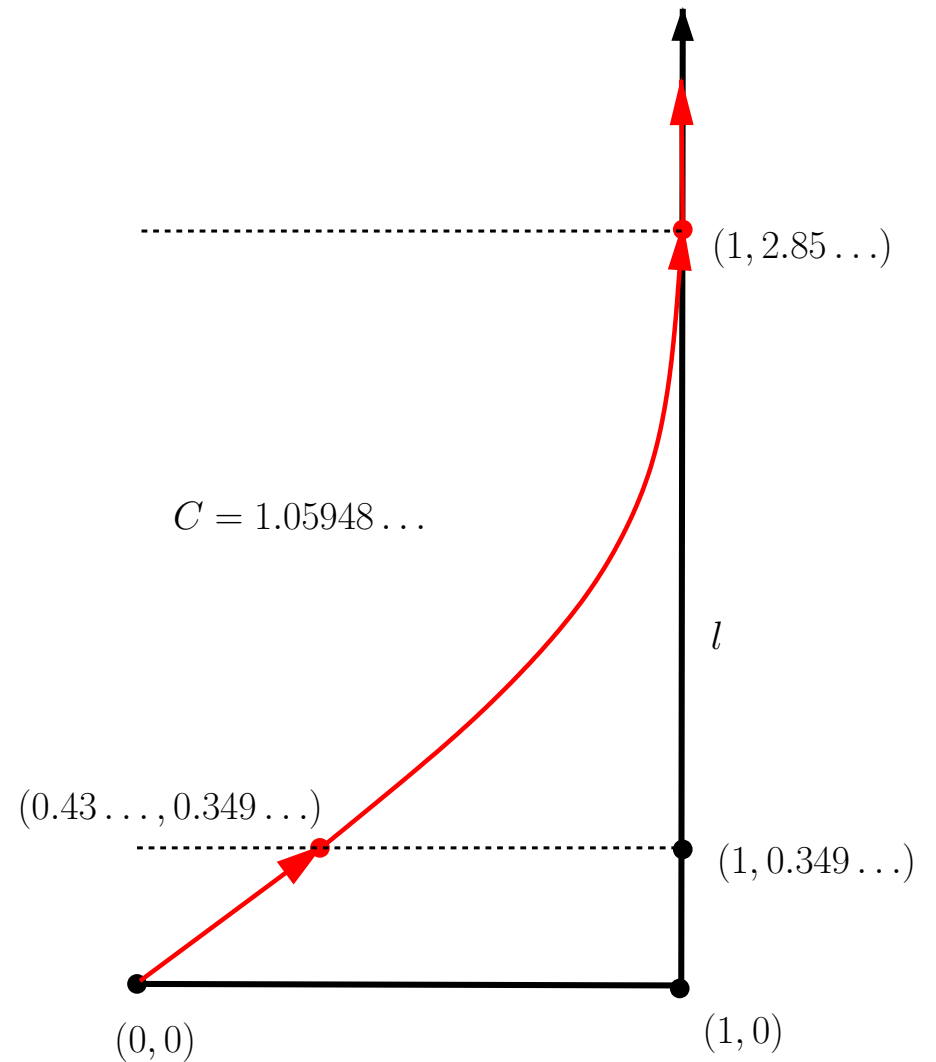
Optimality of f (or f^{-1})

- Solve $f^{-1}(1/b) = 1$: $b = 0.3497\dots$, $D = 1/b = 2.859\dots$,
- $a = 0.43\dots$, worst-case ratio $C = \sqrt{1 + b^2} = 1.05948\dots$
- f convex from (a, b) to $(1, D)$, line segment convex
- Prolongation of line segment is tangent of f^{-1} at (b, a)
- Insert: $f^{-1'}(b) = \frac{a}{b}$



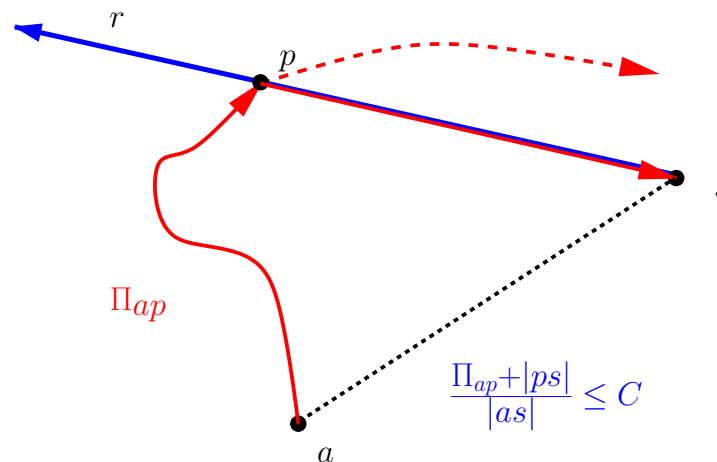
Conclusion

- Optimal strategy with ratio
 $C = 1.05948 \dots$



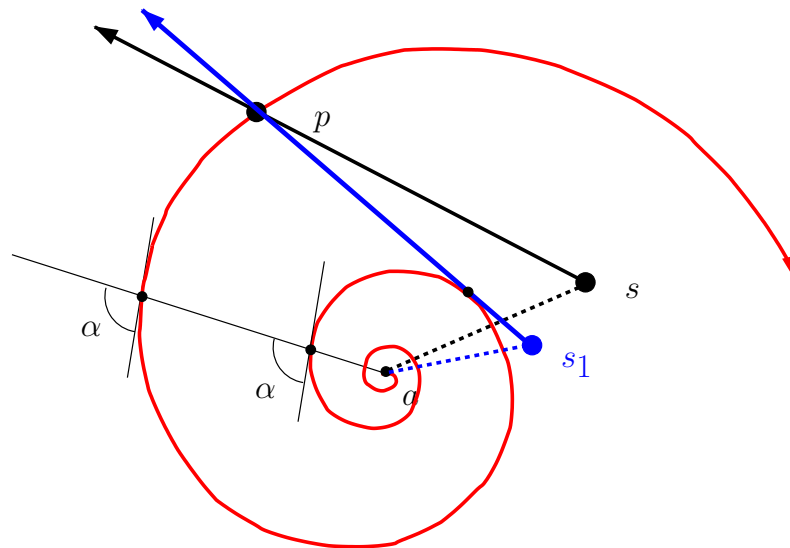
Rays in general

- Rays are somewhere in the plane
- Searchpath Π
- Upper bound: $C = 22.531 \dots$
- Lower bound: $C \geq 2\pi e = 17.079 \dots$



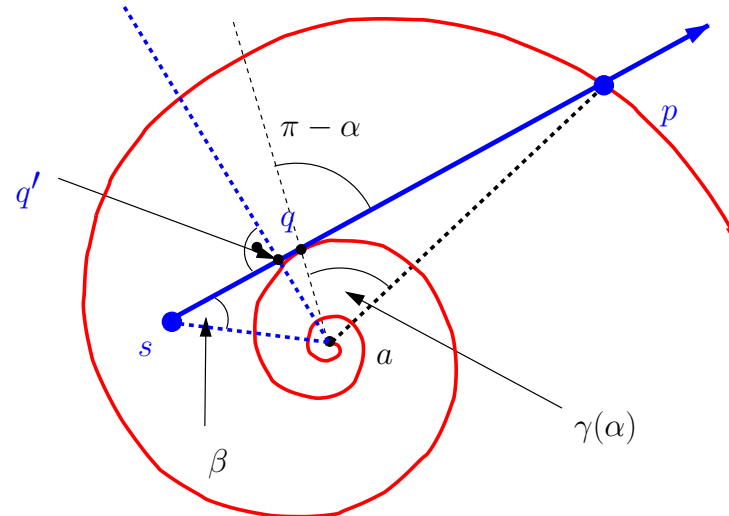
Strategy: Spiral search

- Logarithmic spiral
- Polar coordinates $(\varphi, d \cdot E^{\phi \cot(\alpha)})$, $d > 0, -\infty < \varphi < \infty$
- $\alpha = \pi/2$ Kreis!
- Hits the ray, moves to s
- Worst-case ratio: Ray is a tangent **Lemma**



Optimizing the spiral

- Strategy $d \cdot E^{\phi \cot(\alpha)}$, property $|\text{SP}_a^p| = \frac{|ap|}{|\cos(\alpha)|} = \frac{dE^{\theta_p \cot \alpha}}{|\cos(\alpha)|}$ ■
- Ratio C identical for all tangents: Ratio $C(\alpha)$ ■
- We optimize for perpendicular points q' ■
- Adversary can move s a bit to the left (chooses β) ■
- Law of sine: Ratio $C(\beta, \alpha) = C(\alpha) \sin(\beta) + \cos(\beta)$ ■



Minimize worst-case ratio

- $dE^{\phi \cot(\alpha)}$: Determine α , $d = 1$ ■
- Assume: Fixed α for q' ■
- $|as| \sin(\beta) = |aq'|$,
 $|sq'| = |as| \cos(\beta) = \frac{|aq'| \cos(\beta)}{\sin(\beta)}$ ■
- Ratio $C_{q'} = \frac{|SP_a^p| + |pq'|}{|aq'|}$
 maximized by

$$\frac{C_{q'} \cdot |aq'| + |aq'| \cos(\beta) / \sin(\beta)}{|aq'| / \sin(\beta)} =$$

$$C_{q'} \sin(\beta) + \cos(\beta)$$
 ■
- Minimize over α for q' ■
- Finally adversary choose β , fixes s ! ■

