Online Motion Planning MA-INF 1314
Alternative cost measures!

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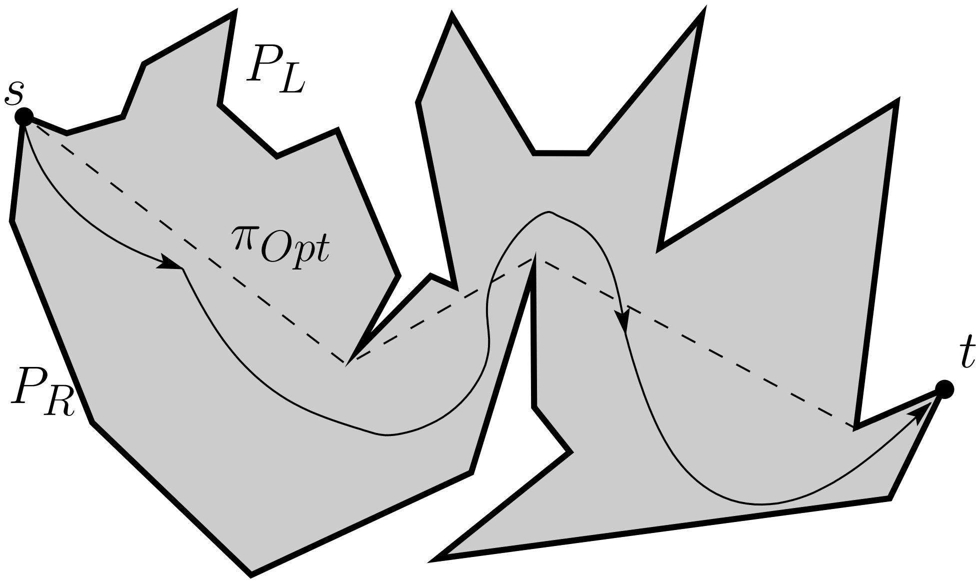
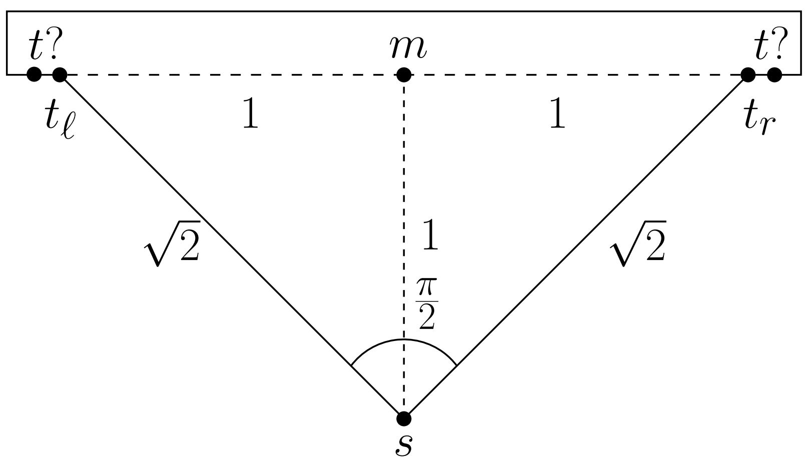
Rep: Searching for target of a street



Rep: Lower Bound

Theorem No strategy can achieve a path length smaller than $\sqrt{2} \times \pi_{\text{Opt}}$.

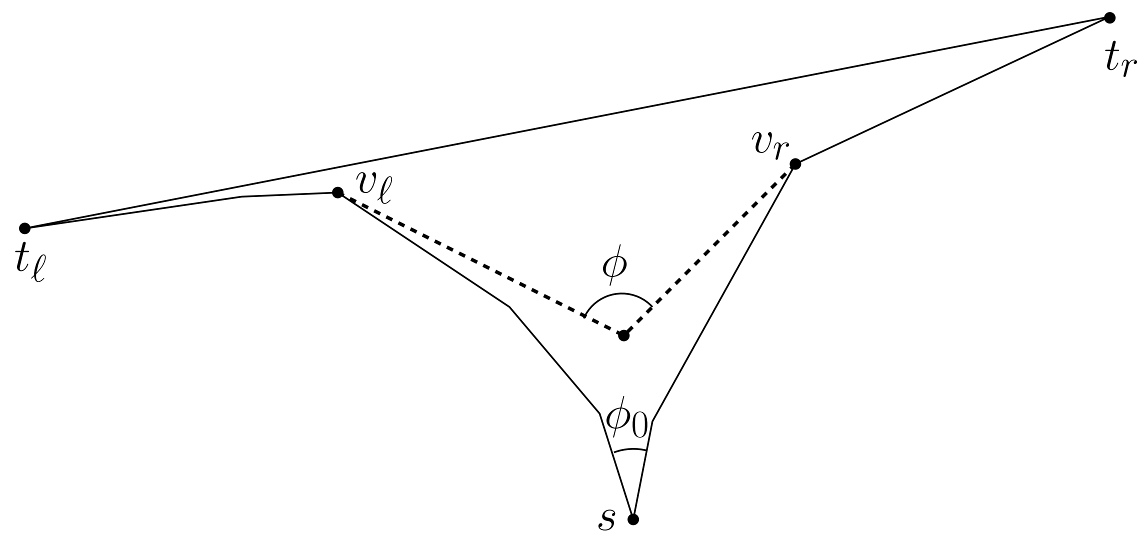
Proof:



Detour with ratio$\sqrt{2}$

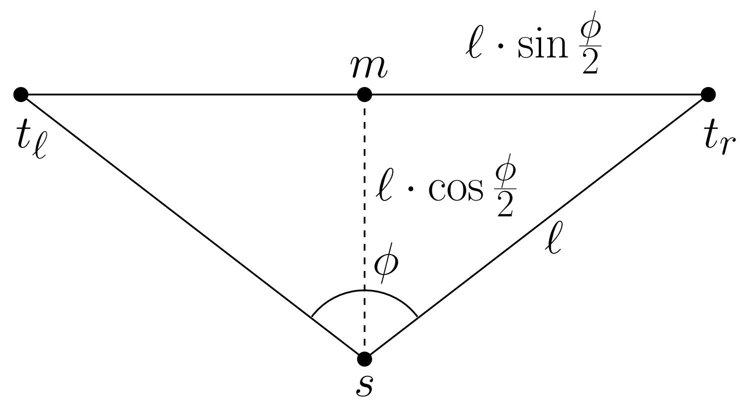
Rep.: Funnel situation!

- It is sufficient to consider special streets only!
- Combine them piecewise!
- **Def.** A polygon that start with a convex vertex s and consists of two opening convex chains ending at t_ℓ and t_r respectively and which are finally connected by a line segment $\overline{t_\ell t_r}$ is called a *funnel* (polygon).



Rep.: Generalized Lower Bound

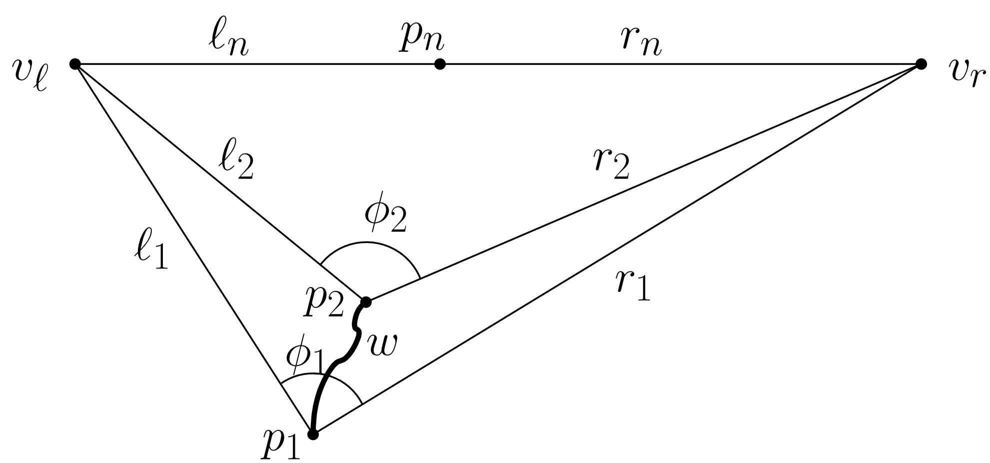
Lemma For a funnel with opening angle $\phi \leq \pi$ no strategy can guarantee a path length smaller than $K_\phi \cdot |Opt|$ where $K_\phi := \sqrt{1+\sin\phi}$. Proof:



Detour at least: $\frac{|\pi_S|}{|\pi_{Opt}|} = \frac{\ell\cos\frac{\phi}{2}+\ell\sin\frac{\phi}{2}}{\ell} = \sqrt{1+\sin\phi}$.

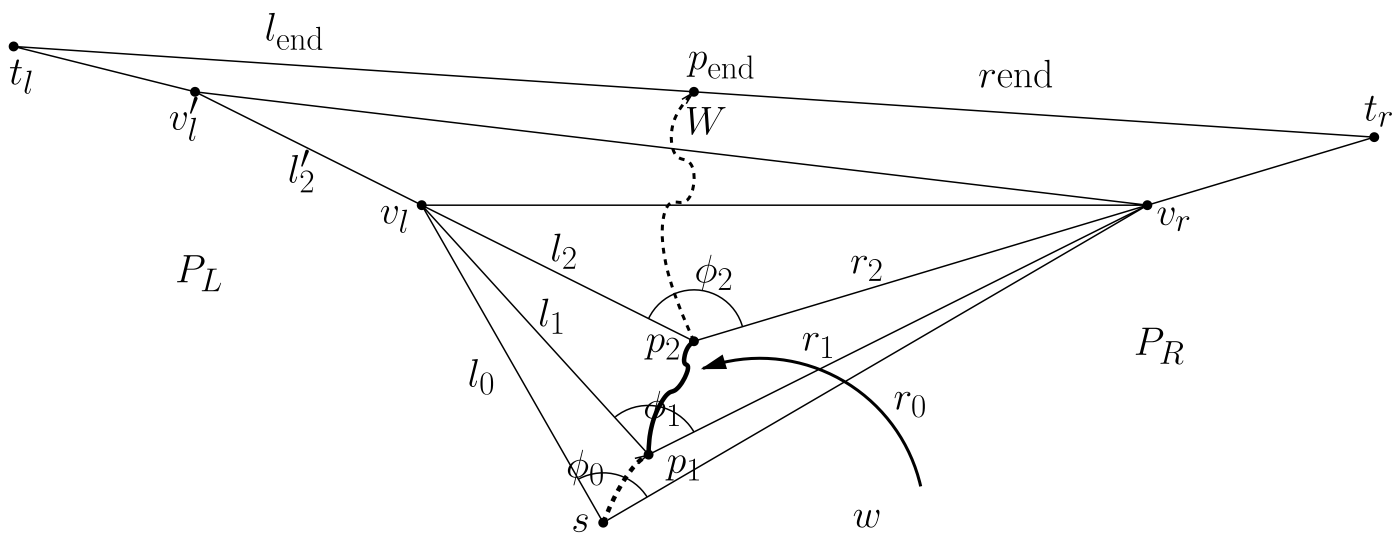
Rep.: Opening angle $\pi \geq \varphi_0 \geq \pi/2$!

- Backward analysis!
- $\frac{|w|+K_{\phi_2}\cdot\ell_2}{l_1} \leq K_{\phi_1}$ and $\frac{|w|+K_{\phi_2}\cdot r_2}{r_1} \leq K_{\phi_1}$
- Combine to one condition for w
- $|w| \leq \min\{\, K_{\phi_1}\ell_1 - K_{\phi_2}\ell_2 \,,\; K_{\phi_1} r_1 - K_{\phi_2} r_2 \,\}$



Rep.: Opening angle $\pi \geq \varphi_0 \geq \pi/2$!

- Change of the reflex vertices! Sufficient!
- Change left side! Condition:
  $|w| \leq \min\{ K_{\phi_1}\ell_1 - K_{\phi_2}\ell_2 \,,\, K_{\phi_1}r_1 - K_{\phi_2}r_2 \}$
- And also: $\frac{|w|+K_{\phi_2}\cdot(\ell_2+\ell_2')}{(l_1+\ell_2')} \leq K_{\phi_1}$

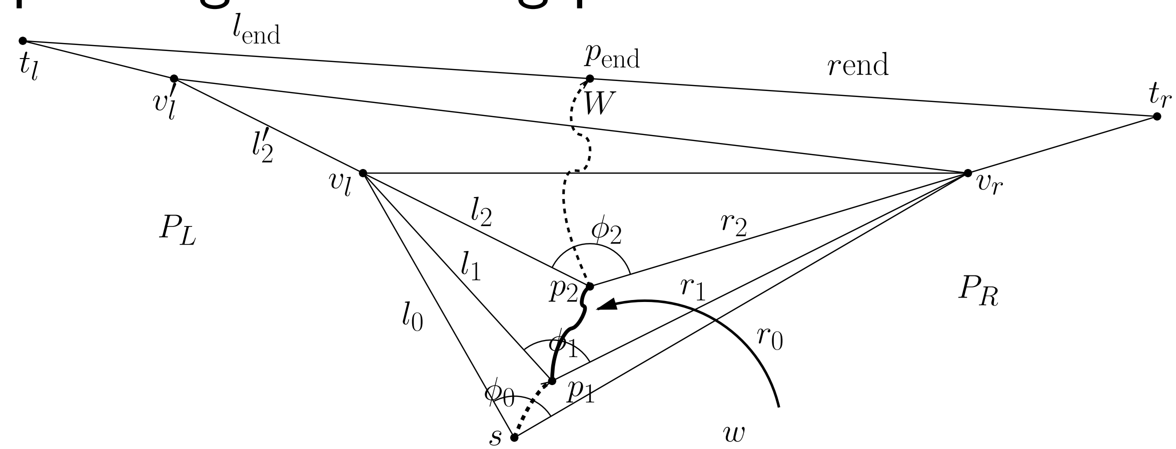


Rep.: Opening angle $\pi \geq \varphi_0 \geq \pi/2$!

Lemma: Let S be a strategy, that searches for the target in a funnel with opening angle ϕ_2 for $\phi_2 \geq \frac{\pi}{2}$ with competitive ratio K_{ϕ_2}. This strategy can be extended to a strategy of ratio K_{ϕ_1} and opening angle ϕ_1 for $\phi_2 > \phi_1 \geq \frac{\pi}{2}$, if we guarantee

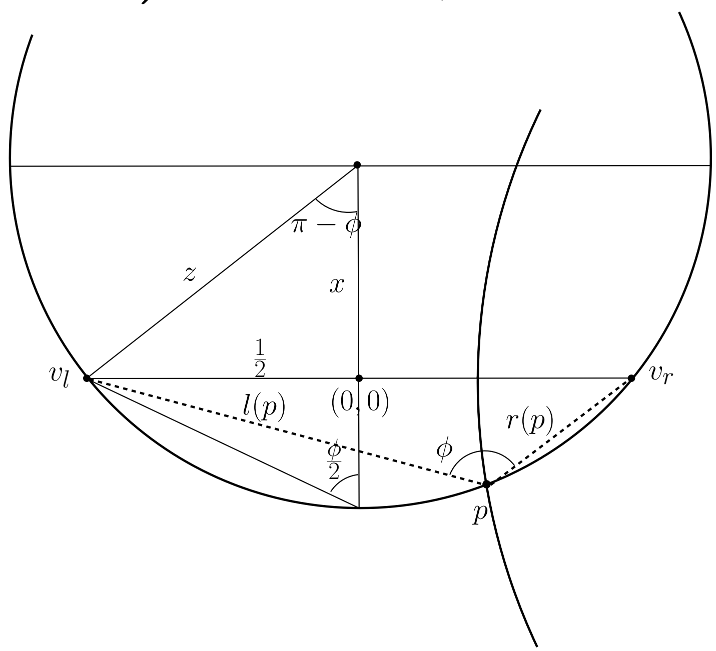
$$|w| \leq \min\{\, K_{\phi_1}\ell_1 - K_{\phi_2}\ell_2 \,,\; K_{\phi_1}r_1 - K_{\phi_2}r_2 \,\}$$

for the corresponding connecting path.



Rep.: Opening angle $\pi \geq \varphi_0 \geq \pi/2$!

- Equality: $K_{\phi_2}(\ell_2 - r_2) = K_{\phi_1}(\ell_1 - r_1)$, $A := K_{\phi_0}(\ell_0 - r_0)$
- Hyperbola: $\dfrac{X^2}{\left(\frac{A}{2K_\phi}\right)^2} - \dfrac{Y^2}{\left(\frac{1}{2}\right)^2 - \left(\frac{A}{2K_\phi}\right)^2} = 1$
- Circle: $X^2 + \left(Y + \dfrac{\cot\phi}{2}\right)^2 = \dfrac{1}{4\sin^2\phi}$

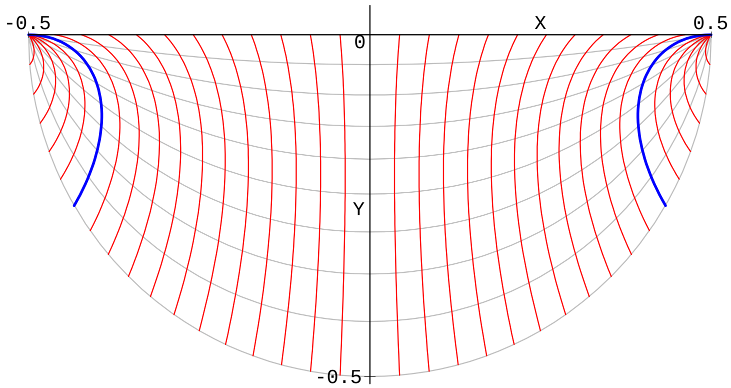


Rep.: Opening angle $\pi \geq \varphi_0 \geq \pi/2$!

Intersection Hyp. Circle: Curve!!

$$
\begin{aligned}
X(\phi) &= \frac{A}{2} \cdot \frac{\cot \frac{\phi}{2}}{1+\sin\phi} \cdot \sqrt{\left(1+\tan\frac{\phi}{2}\right)^2 - A^2} \\
Y(\phi) &= \frac{1}{2} \cdot \cot\frac{\phi}{2} \cdot \left(\frac{A^2}{1+\sin\phi} - 1\right)
\end{aligned}
$$

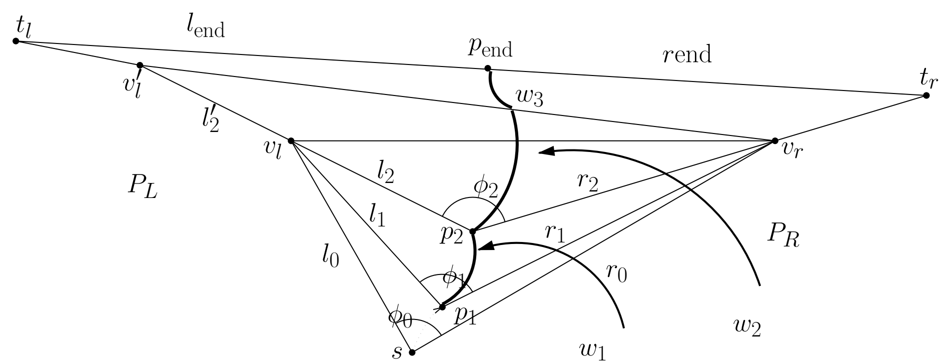
where $A = K_{\phi_0}(\ell_0 - r_0)$



Opt. strat. for opening angle $\pi \geq \varphi_0 \geq \pi/2$!

$$X(\phi) = \frac{A}{2} \cdot \frac{\cot\frac{\phi}{2}}{1+\sin\phi} \cdot \sqrt{\left(1+\tan\frac{\phi}{2}\right)^2 - A^2}$$
$$Y(\phi) = \frac{1}{2} \cdot \cot\frac{\phi}{2} \cdot \left(\frac{A^2}{1+\sin\phi} - 1\right)$$

Change of the boundary points. A also changes, new piece of curve!



Opt. strat. for opening angle $\pi \geq \varphi_0 \geq \pi/2$!

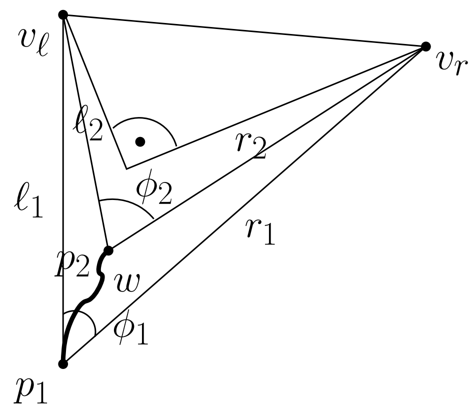
Theorem: The goal of a funnel with opening angle $\phi_0 > \frac{\pi}{2}$ can be found with ratio K_{ϕ_0}.

Proof: Show that the curves fulfil:
$|w| \leq \min\{\, K_{\phi_1}\ell_1 - K_{\phi_2}\ell_2 \,,\; K_{\phi_1}r_1 - K_{\phi_2}r_2 \,\}$

For any small piece w of the curve. Analytically, lengthy proof!
Experimentally!

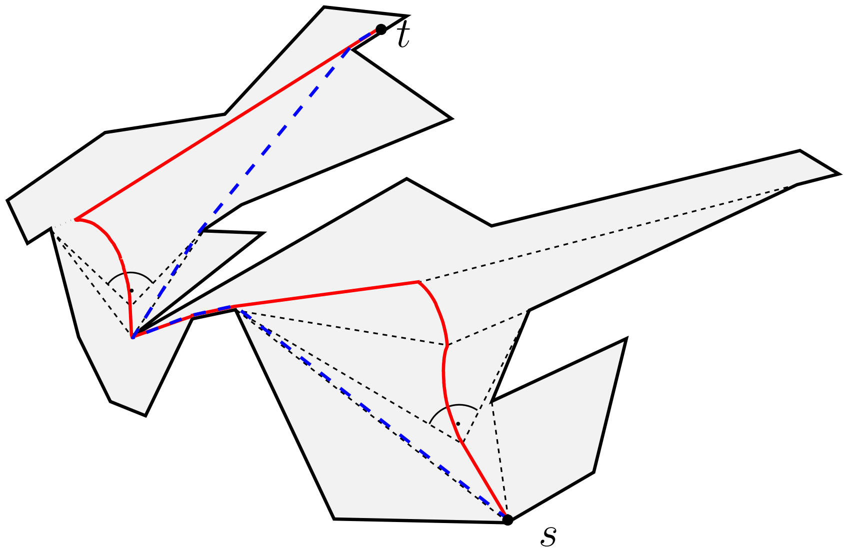
Opt. strat. opening angle $0 \le \varphi_0 \le \pi/2$!

- The same approach
- But independent from the angle
- Dominated by *factor* $K_{\pi/2} = \sqrt{2}$
- Require: $w \le \min\{\, \sqrt{2}(\ell_1 - \ell_2)\ ,\ \sqrt{2}(r_1 - r_2)\,\}$.
- Equality: $\ell_1 - \ell_2 = r_1 - r_2$
- Current angular bisector: Hyberbola!



Opt. strat. opening angle $0 \leq \varphi_0 \leq \pi$!

Combine strategy 1 and strategy 2



Theorem: In an unknown street-polygon beginning from the source s we can find the target t with an optimal online strategy with competitive ratio $\sqrt{2}$.

Optimal strategy "Worst-Case-Aware"

As long as target t is not visible:

Compute current v_ℓ and v_r.

If only one exists: Move directly toward the other.

Otherwise. Repeat:

New reflex vertex v'_ℓ or v'_r is detected:

Use v'_ℓ or v'_r instead of v_ℓ or v_r.

Let ϕ be the angle between v_ℓ, the current poyition and v_r.

If $\phi \leq \frac{\pi}{2}$: Follow the current angular bisctor!

If $\phi > \frac{\pi}{2}$: Follow the curve $(X(\phi), Y(\phi))$.

Until either v_ℓ or v_r is explored.

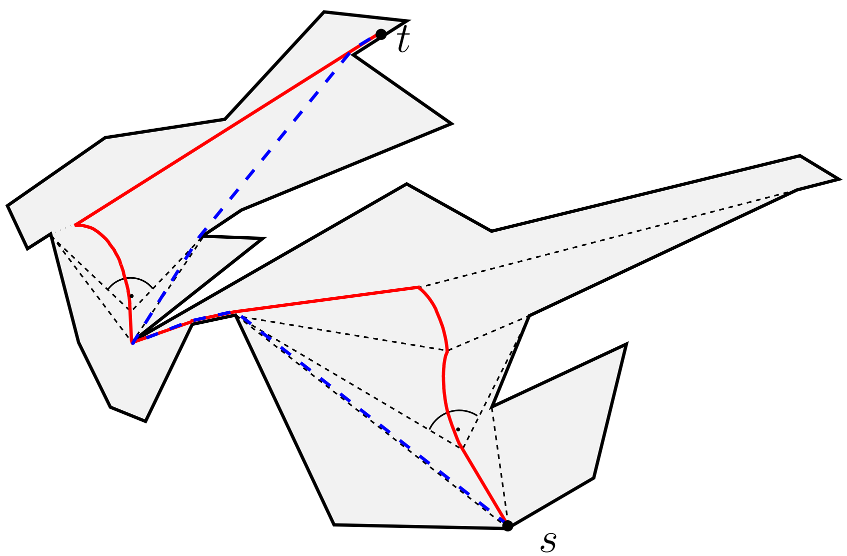
Move toward the non-explored vertex.

Move toward the goal.

Opening angle $\pi \geq \varphi_0 \geq \pi/2$!

Theorem: Searching for the target in a street polygon can be realized within a competitive ratio of $\sqrt{2}$.

- From $\varphi \geq \pi/2$ curve fulfils w condition, analysis/experiments!
- For smaller angles: $\sqrt{2}$ substitute for all K_ϕ



Optimal searchpath

- We have seen:
  Searching for a goal (polygon) in general not competitive
- Question: What is a good searchpath (for polygons)?
- Searching: Target point unknown!
- Offline-Searching: Environment is known
- Online-Searching: Environment unknown

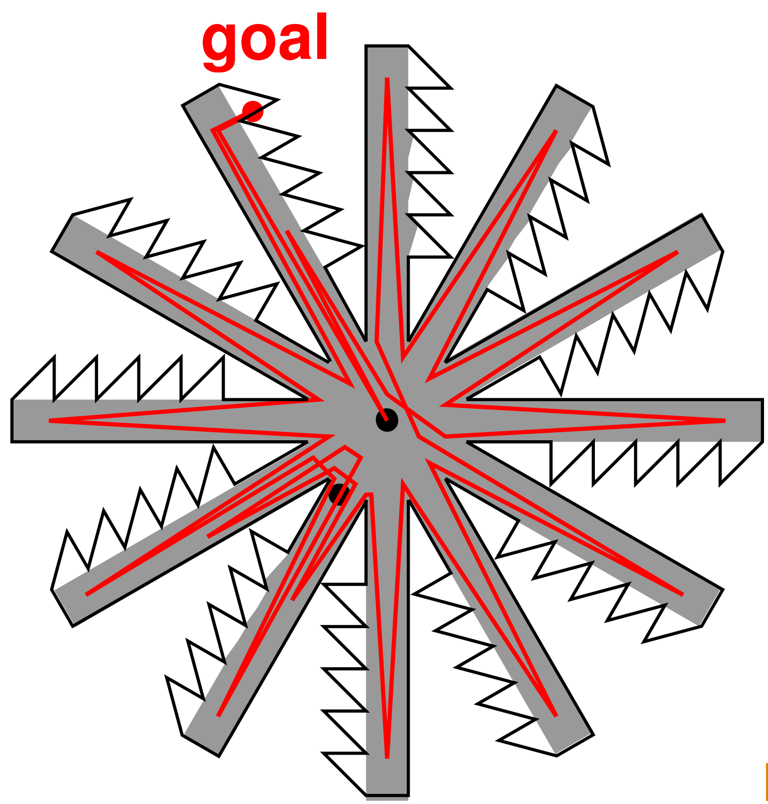
Quality measures!

- *Competitive ratio* of search strategy $\mathcal{A}$ in polygons:

$$C := \sup_P \sup_{p \in P} \frac{|\mathcal{A}(s,p)|}{|\mathsf{sp}(s,p)|}$$

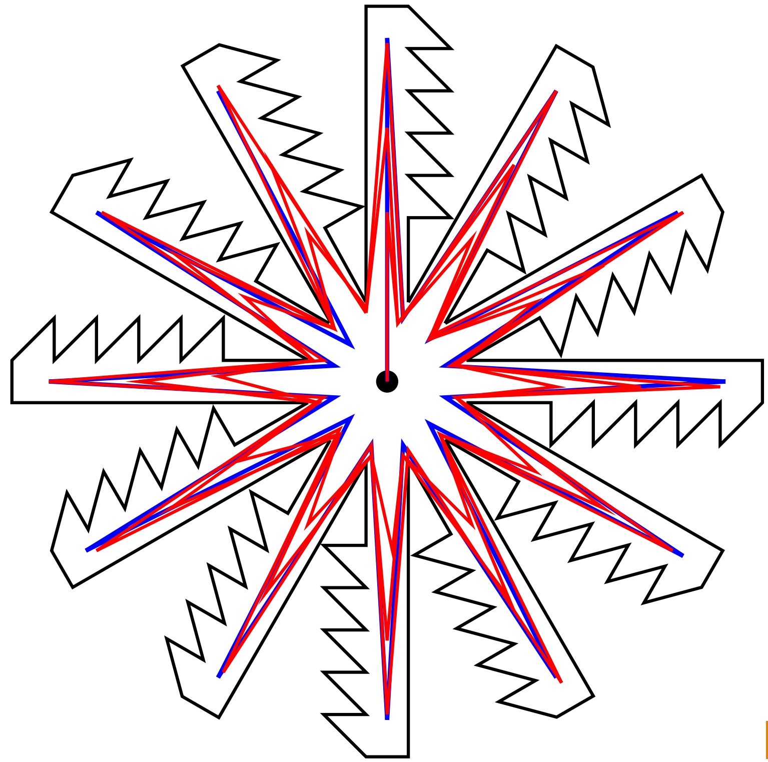
Optimal search path in polygons

- Competitive analysis:
- Agent with visibiltly
- Adversary forces any strategy to visit any corridor
- Optimal path is short
- $\Rightarrow$ Any strategy fails (not constant competitive)



Optimal search path in polygons

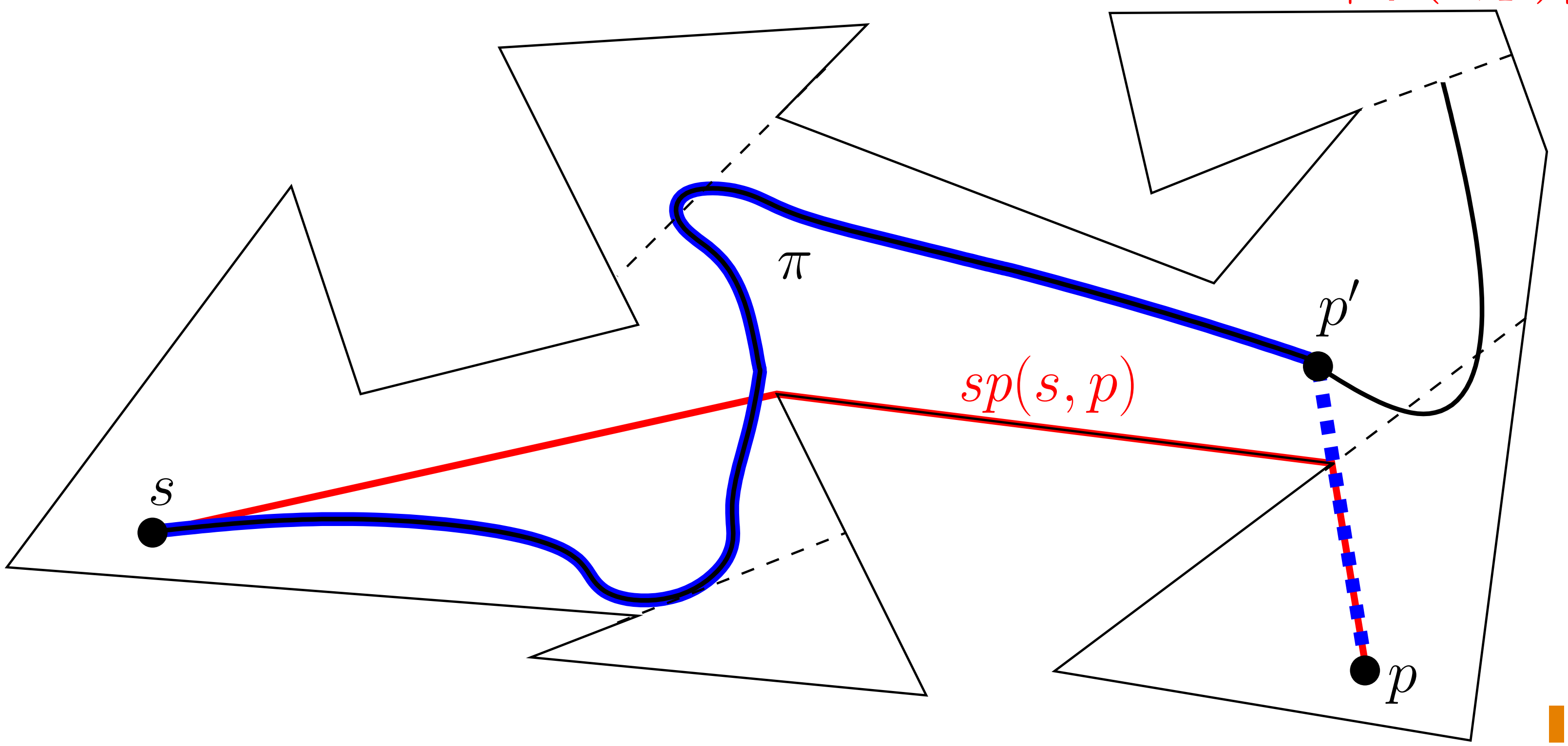
- Strat1:
  fully visit any corridor
- Strat2:
  visit all corr. depth $d = 1$
  visit all corr. depth $d = 2$
  visit all corr. depth $d = 4$ etc.



- Strat2 seems to be better:
  close targets s are visited earlier
- Can we give a measure?

Search ratio for polygons

π: **Searchpath**, quality for π: $\mathrm{SR}(\pi, P) = \max\limits_{p \in P} \dfrac{|\pi_s^{p'}| + |p'p|}{|\mathrm{sp}(s,p)|}$



Search ratio in general

Given: Environment $\mathcal{E}$, Set of goals $\mathcal{G} \subseteq \mathcal{E}$

Graphs $G = (V, E)$: **Vertices** $\mathcal{G} = V$

Geometric Search $\mathcal{G} = V \cup E$

(Requirement: $\forall p \in \mathcal{E} : |\mathsf{sp}(s, p)| = |\mathsf{sp}(p, s)|$)

Search ratio of a search strategy $\mathcal{A}$ for $\mathcal{E}$:

$$\mathsf{SR}(\mathcal{A}, \mathcal{E}) := \sup_{p \in \mathcal{G}} \frac{|\mathcal{A}(s, p)|}{|\mathsf{sp}(s, p)|}$$

Optimal search ratio:

$$\mathsf{SR}_{\mathsf{OPT}}(\mathcal{E}) := \inf_{\mathcal{A}} \mathsf{SR}(\mathcal{A}, \mathcal{E})$$

Path with optimal search ratio

- Graphs (offline): NP-hard
- Polygons (offline): ???
- Online: Approximation is possible

$\Rightarrow$ Goal: Approximate the path with opt. search ratio

Search ratio approximation

- *Competitive ratio :* $C := \sup_{\mathcal{E}} \sup_{p \in \mathcal{G}} \frac{|\mathcal{A}(s,p)|}{|\mathsf{sp}(s,p)|}$

- *Search ratio:* $\mathsf{SR}(\mathcal{A},\mathcal{E}) := \sup_{p \in \mathcal{G}} \frac{|\mathcal{A}(s,p)|}{|\mathsf{sp}(s,p)|}$

- *Optimal search ratio*: $\mathsf{SR}_{\mathsf{OPT}}(\mathcal{E}) := \inf_{\mathcal{A}} \mathsf{SR}(\mathcal{A},\mathcal{E})$

- Approximation: $\mathcal{A}$ *search-competitiv*

$$C_s := \sup_{\mathcal{E}} \frac{\mathsf{SR}(\mathcal{A},\mathcal{E})}{\mathsf{SR}_{\mathsf{OPT}}(\mathcal{E})}$$

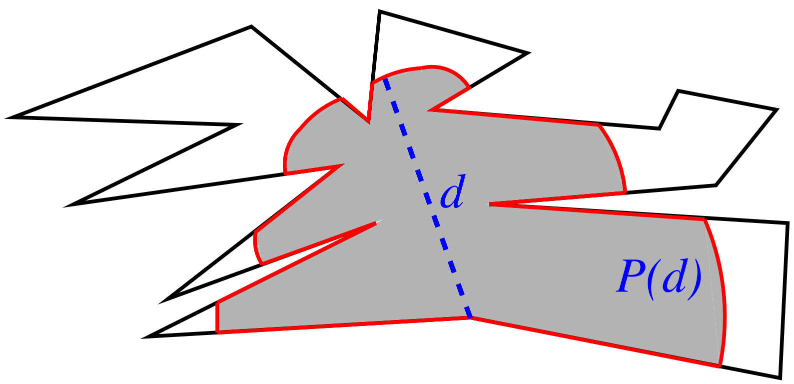
- Comparison not against SP, but against best possible SR

Depth-restricted exploration

Def. Exploration-Strategy Expl for $\mathcal{E}$ is called **depth-restrictable**, if we can derive a strategy $\mathrm{Expl}\,(d)$ such that:

- $\mathrm{Expl}\,(d)$ explores $\mathcal{E}$ up to depth $d \geq 1$
- return to s after the exploration
- $\mathrm{Expl}\,(d)$ is C-competitive, i.e.,$\exists\, C \geq 1 :\ \forall \mathcal{E}$:

$$|\mathrm{Expl}\,(d)| \leq C \cdot |\mathrm{Expl}_{\mathrm{OPT}}(d)|\,.$$



Searchpath approximation

Algorithm

- Explore $\mathcal{E}$ by increasing depth: $\mathrm{Expl}\,(2^i)$ für $i = 1, 2, \ldots$

Lemma:

- Agent without vision system
- Environment $\mathcal{E}$
- $\mathrm{Expl}_{\mathrm{ONL}}$: C-competitive, depth-restrictable, online exploration strategy for $\mathcal{E}$
  (d. h. $|\mathrm{Expl}\,(d)| \leq C \cdot |\mathrm{Expl}_{\mathrm{OPT}}(d)|$)

$\Rightarrow$ Algorithm gives $4C$-Approximation of optimal search path!

Searchpath approximation proof

$$|\Pi_{\mathrm{Expl}_{\mathrm{opt}}(d)}| \leq d \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1) \tag{1}$$

$$\begin{aligned}
\mathsf{SR}(\Pi) \quad & \leq \quad \frac{\sum_{i=1}^{j+1} |\Pi_{\mathrm{Expl}(2^i)}|}{2^j + \varepsilon} \\
& \underset{(\mathsf{Ass.})}{\leq} \quad \frac{C}{2^j} \sum_{i=1}^{j+1} |\Pi_{\mathrm{Expl}_{\mathrm{opt}}(2^i)}| \underset{(1)}{\leq} \frac{C}{2^j} \sum_{i=1}^{j+1} 2^i \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1) \\
& \leq \quad C \cdot \left(\frac{2^{j+2} - 1}{2^j} \right) \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1) \leq 4C \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1)
\end{aligned}$$

Applications

- **Corollay:**
  Trees: Exploration by DFS ($C = 1$) Online or Offline, depth restricted, simple

$\Rightarrow$ Searchpath approximation of factor 4

- Graphs: Online and Offline!
  CFS ($C = 4 + \frac{8}{\alpha}$) depth-restrictable!!
- But: Factor depends on rope length $(1 + \alpha)r$ by depth r
- CFS sometimes explores more than d (precisely $(1 + \alpha)d$)

$\Rightarrow$ $\mathrm{Expl}\,(d)$ not comparable to $\mathrm{Expl}_{\mathrm{OPT}}(d)$

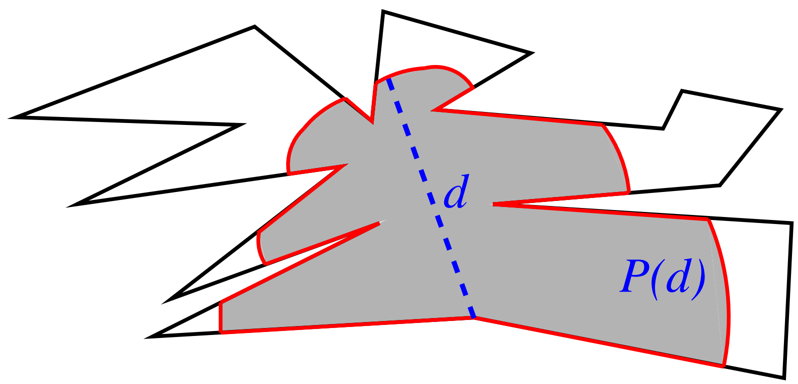
- Workaround: Compare $\mathrm{Expl}\,(d)$ with $\mathrm{Expl}_{\mathrm{OPT}}(\beta \cdot d)$

β-depth restricted exploration

Def. Exploration strategy Expl for $\mathcal{E}$ is denoted as **β-depth restrictable**, if we can derive a strategy $\mathrm{Expl}\,(d)$ such that:

- $\mathrm{Expl}\,(d)$ explores $\mathcal{E}$ only up to depth $d \geq 1$
- returns to the start s
- $\mathrm{Expl}\,(d)$ is C_β-competitiv, i.e., $\exists\, C_\beta \geq 1, \beta > 0 : \forall \mathcal{E}$:

$$|\mathrm{Expl}\,(d)| \leq C_\beta \cdot |\mathrm{Expl}_{\mathrm{OPT}}(\beta \cdot d)|\,.$$



Searchpath approximation

Theorem:

- Agent without vision
- Environment $\mathcal{E}$
- $\mathrm{Expl}:$ C_β-competitive, β-depth restrictable, online exploration strategy for $\mathcal{E}$, (i.e., $|\mathrm{Expl}\,(d)| \leq C_\beta \cdot |\mathrm{Expl}_{\mathrm{OPT}}(\beta \cdot d)|$)

$\Rightarrow$ Algorithm (exploration/double depth) gives a $4\beta C_\beta$-approximation of the optimal searchpath ▮

Corollary: Unknown graphs, Algorithm with CFS is $4(1+\alpha)(4+\frac{8}{\alpha})$-approximation of optimal searchpath ▮