Online Motion Planning MA-INF 1314
Polygon/Corner Exploration

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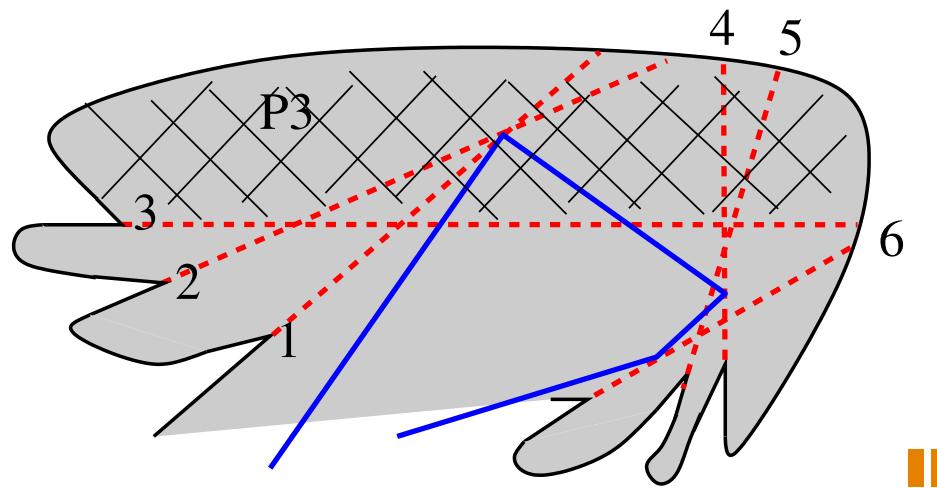
Rep.: Search Path Approx. Applications (vision)!

- Simple polygon, Offline: SWR ($C_\beta = 1 = \beta$)
  $\Rightarrow$ 8-Approximation
- Rectilinear Polygons, Online: Greedy-Online ($C_\beta = \sqrt{2}, \beta = 1$)
  $\Rightarrow 8\sqrt{2}$-Approximation
- Simple Polygons, Offline: Polytime Alg. ($C_\beta = 1, \beta = 1$)
  $\Rightarrow$ 8-Approximation
- Simple Polygons, Online: PolyExplore ($C_\beta = 26, \beta = 1$)
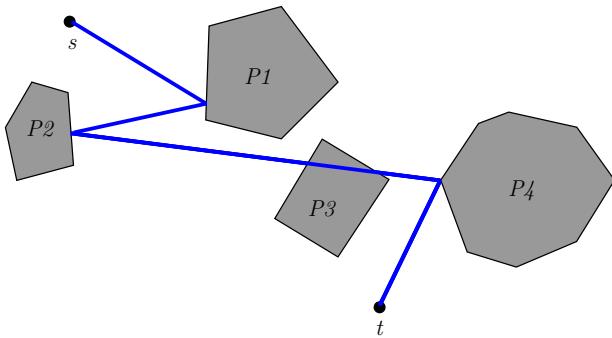 $\Rightarrow$ 212-Approximation

Consider exploration task! Full and depth restricted!

Rep.: SWR (General case): Offline!

- Corner problem!!■
- Sequence of essential cuts, successive cuts■
- Not visited by order along boundary. ■
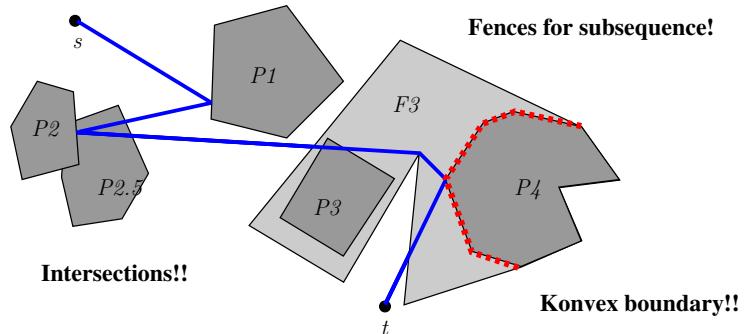- But the corresponding P_{c_i}!!!■
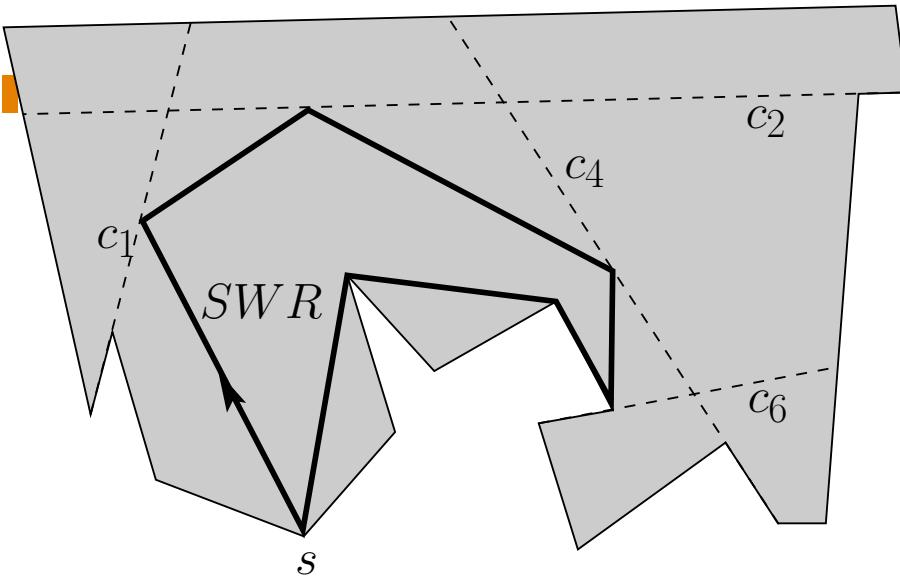
Rep.: Touring a sequence of polygons (TPP)

- Simple version:
- $O(nk \log \frac{n}{k})$
- Build(Query): $O(nk \log \frac{n}{k})$
- Compl.: $O(n)$
- Query (fixed s): $O(k \log \frac{n}{k})$

- General version:
- Fences, convex boundary, etc.
- $O(nk^2 \log n)$
- Build(Query): $O(nk^2 \log n)$
- Compl.: $O(nk)$
- Query (fixed s): $O(k \log n + m)$

Results from: Dror, Efrat, Lubiw, Mitchell 2003!!
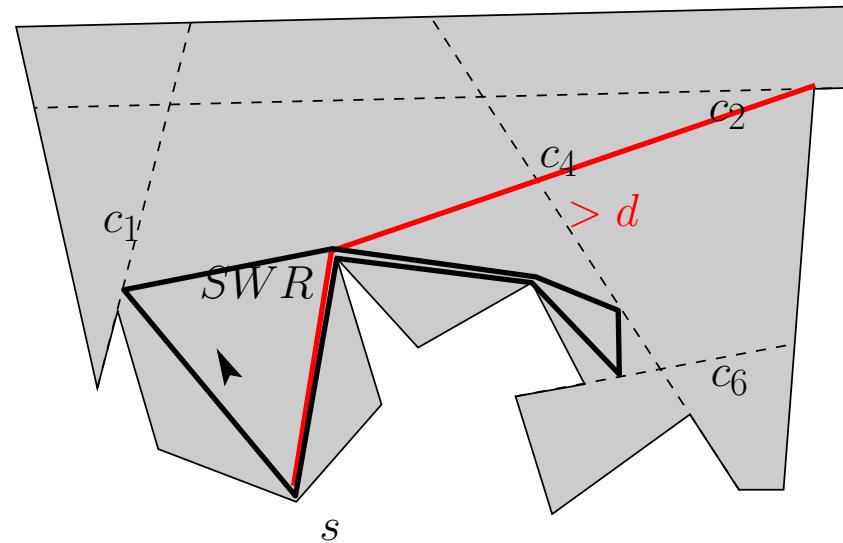
Application: SWR

Essential *parts*! Use the order along the boundary!

One common fence, intersections!

Start and target identical!

- $O(n^4)$ '91
- $O(n^4)$ Tan et al. '99
- $O(n^3 \log n)$ by this result!
- **Theorem**

Rep.: Application: General simple polygons Offline

- Compute optimal exploration tour
- Agent with vision, start s at the boundary
- Depth restriction: Ignore cuts with distance $> d$
- $\mathrm{Expl}(d) = \mathrm{Expl}_{\mathrm{OPT}}(d)$
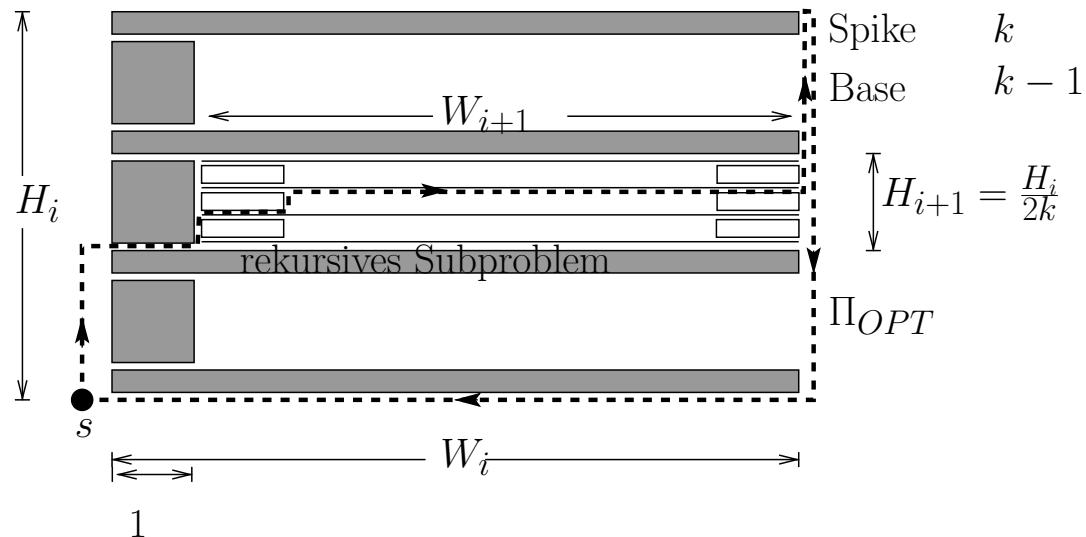- **Theorem**: 8 Approximation, Online??

Rep.: Polygons with holes

There is no constant online approximation of the optimal search ratio

Theorem Let A be an online strategy for the exploration of a polygon with n obstacles (holes), we have: $|\Pi_A| \geq \sqrt{n}|\Pi_{OPT}|$
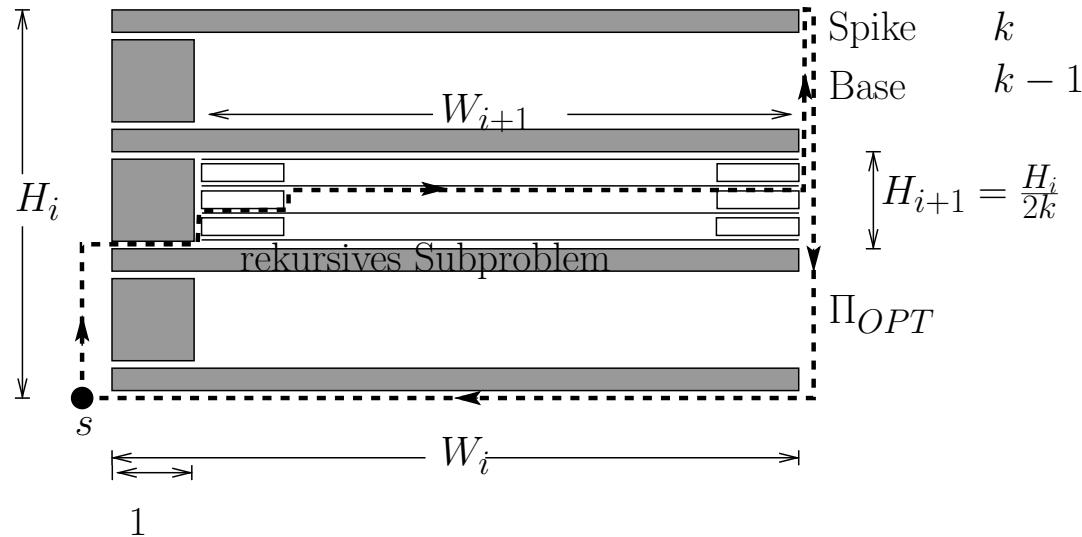
Proof: LB by examples!

Rep.: Polygon with holes: $|\Pi_A| \geq \sqrt{n}|\Pi_{OPT}|$

- $W_1 = 2k$, $H_1 = k$, k spikes, $(k-1)$ bases, $(2k-1)k$ rectangles▮
- $H_i = \frac{H_1}{(2k)^{i-1}}$, $W_i = 2k - i + 1 \geq k$, $i = 1, \ldots, k$▮
- Situation H_i: Online strategy does not know position of block H_{i+1}▮

- Left side: Look behind any block
- Right side: Move once upwards▮
- Adversary: Find block after $\Omega(k)$ steps▮
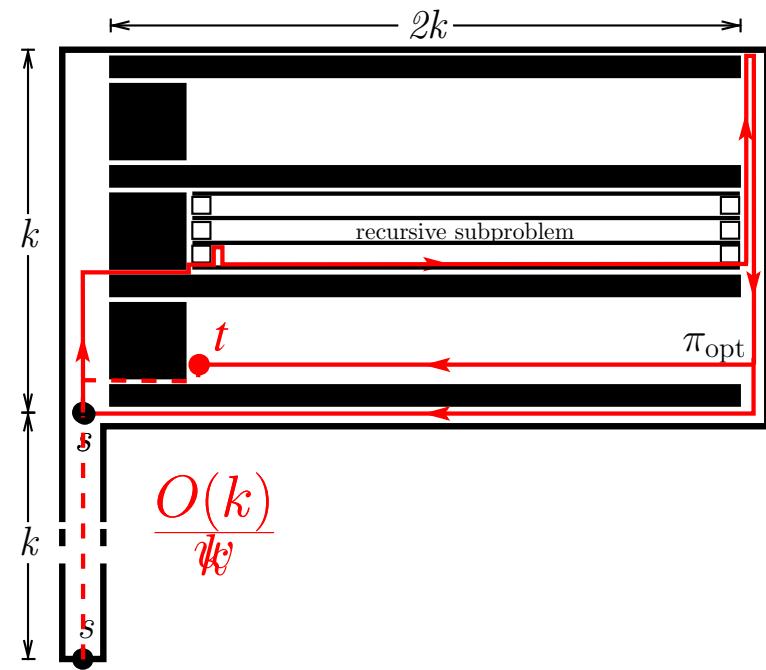- Altogehter: $\Omega(k \times k)$ for any Strategy ▮

Rep.: Polygons with holes: $|\Pi_A| \geq \sqrt{n}|\Pi_{OPT}|$

- Optimale strategy: Move directly to the block
- Go on recursively, at the end move along any block
- $[\Pi_{OPT}| = W_1 + 2\sum_{i=1}^{k} H_i \leq 6k$
- $k = \lfloor\sqrt{n}\rfloor$ gives the result

Rep.: Polygons with holes Corollary

- No $O(1)$-competitive exploration for such environments $(\Omega(\sqrt{n}))$
- Optimal exploration has a bad Search Ratio
- Trick: Extension
- Then: Optimal exploration has Search Ratio $O(1)$
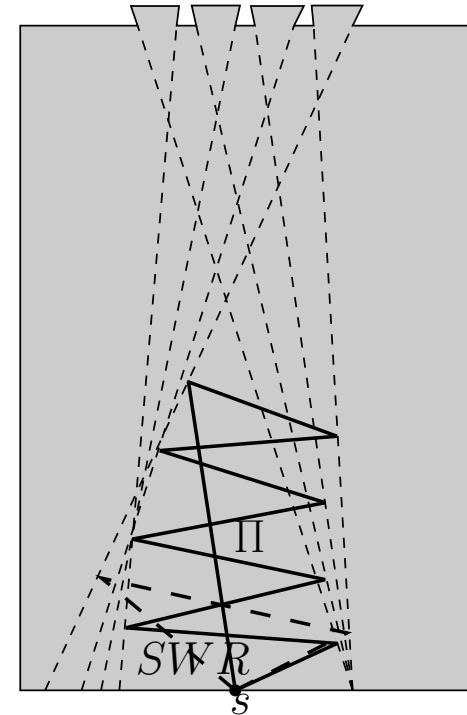- Any online strategy has Search Ratio $\Omega(k)$

Rep.: Summary

- Connection between exploration and search:

- $\exists$ constant-competitive, depth-restrictable exploration strategy
  $\Rightarrow$ $\exists$ search strategy with mit competitiver Search Ratio

- $\nexists$ constant-competitive exploration strategy,
  but $\exists$ 'extendable' lower bound
  $\Rightarrow$ $\nexists$ search strategy with competitive Search Ratio
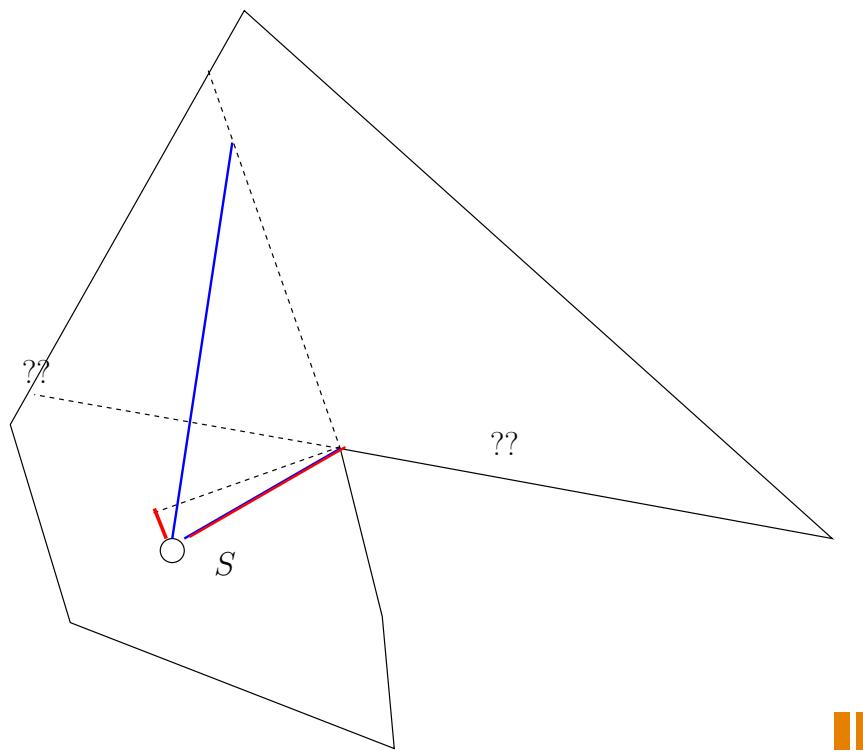
Rep.: Online exploration general polygon

- Essential cuts by order along boundary▮
- Not competitive▮
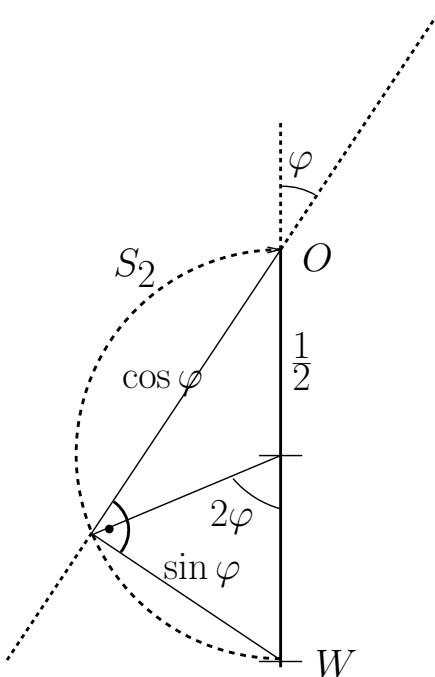- Subdivision into: Right and Left reflex vertices▮

Rep.: Online exploration general polygons

- Subtask: Explore a single corner ▮
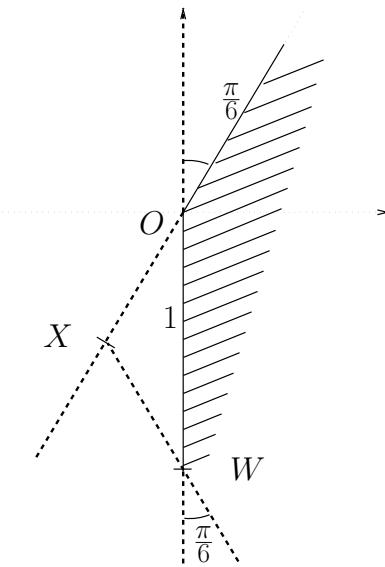- Looking around a corner ▮
- Simple strategies: Circular arcs! ▮

Rep.: Looking around a corner, Theo.

- $H(\varphi) = \varphi$, monotonically increasing: Maximum for $\varphi = \frac{\pi}{2}$
- $G(\varphi) = \frac{\varphi}{\sin \varphi}$, $G'(\varphi) = \frac{\sin \varphi - \varphi \cos \varphi}{\sin^2 \varphi}$
- Optimize: $G'(\varphi) > 0$ für $\varphi \in (0, \pi/2]$, Max. for $\varphi = \frac{\pi}{2}$
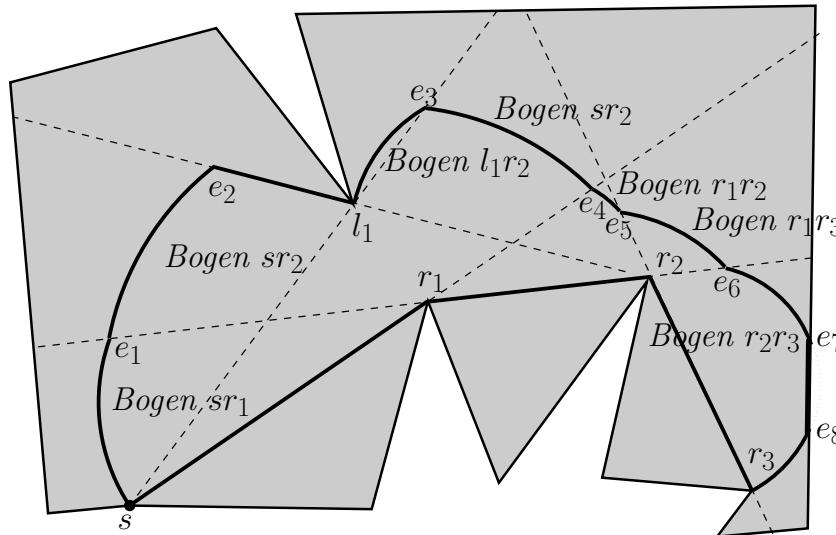- Ratio: $\frac{\pi}{2}$ because pathlength $\pi/2 \times 1$

Looking around a corner: Simple lower bound!

- Special situation■
- Strategy move left or right w.r.t. X!■
- In any case: Ratio $\frac{2}{3}\sqrt{3} = \frac{2}{\sqrt{3}}$, Blackboard! **Theo.** ■
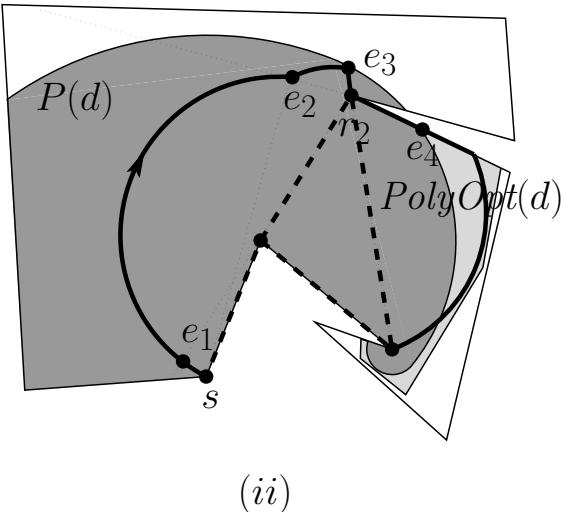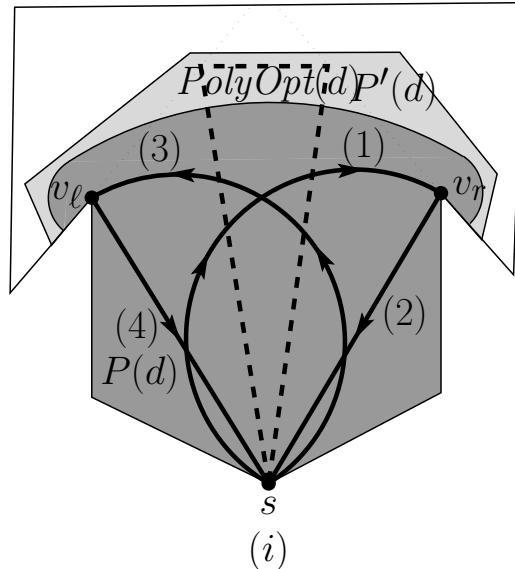- Optimal strategy: ratio $1.212\ldots$ **Theo. (later)**■

Online exploration general polygon

- Exploration by circular arcs▮
- Applet!▮
- **Theorem** A simple polygon can be explored online within a competitive ratio of 26.5.▮
- Depth restrictable: $P(d)$, $\beta = 1$▮

PolyExplore: Depth restricted exploration!

- Up to depth d, Beispiel! ▮
- Ignore the cuts with distance $> d$, leave $P(d)$! Both!▮
- Analysis still holds! $\beta = 1$, $C_\beta = 26.5$▮
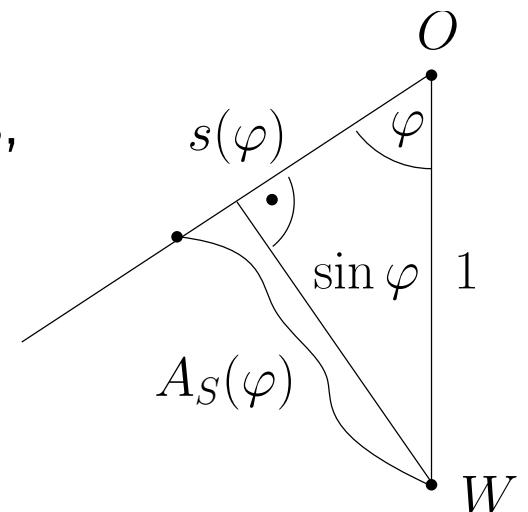- **Theorem** 26.5×8 Approximation of the optimal search path!▮

Look around a corner revisit: Optimal solution!

- Offline strategy: $d(\varphi)$ distance from W to extension $E(\varphi)$ at O

$$d(\varphi) = \begin{cases} \sin\varphi & : & \varphi \in [0, \pi/2] \\ 1 & : & \varphi \in [\pi/2, \pi] \end{cases}$$
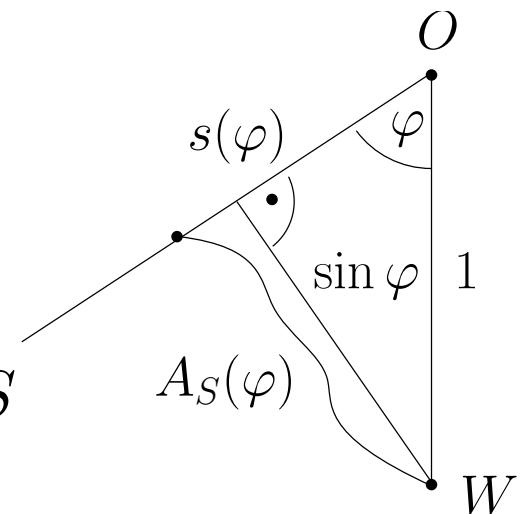
- Online strategy: $S(\varphi, s(\varphi))$ polar coordinates, $\varphi \in [0, \gamma]$, start $(0, -1)$, vertex $O = (0, 0)$.

1. Start at W: $s(0) = 1$

2. s continuous, no jumps, differentiable in $(0, \gamma)$

3. $s(\gamma) \neq 0 \Rightarrow \gamma = \pi$
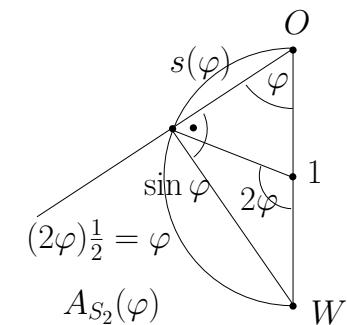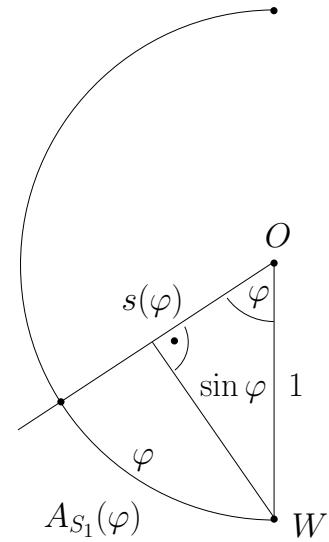
4. $s'(0) < \infty$ (makes sense)

Look around a corner revisit: Optimal solution!

- Online strategy: $S(\varphi, s(\varphi))$ polar coordinates
- Path length: $A_S(\varphi)$ by $s(\varphi)$
- $A_S(\varphi) := \int_0^\varphi \sqrt{s'(t)^2 + s(t)^2} \, dt$
- Ratio: $R_S(\varphi) := \frac{A_S(\varphi)}{d(\varphi)}$
- S is c-competitive if:
  $R_S(\varphi) \leq c$ for all $\varphi \in [0, \gamma]$
- $c_s := \sup_{\varphi \in [0,\gamma]} R_S(\varphi)$ competitive ratio of S
- **Lemma:** S competitive $\Longleftrightarrow |s'(0)| < \infty$
- Proof: Blackboard!
- Almost any strategy is successful!

Examples!

- $S_1(\varphi)$ half-circle with radius 1 around O
- $S_2(\varphi)$ half-circle with radius $\frac{1}{2}$ around midpoint of OW

- $R_{S_1}(\varphi) = \begin{cases} \dfrac{\varphi}{\sin \varphi} & : \quad \varphi \in [0, \pi/2] \\ \varphi & : \quad \varphi \in [\pi/2, \pi] \end{cases}$

- $c_{S_1} = R_{S_1}(\pi) = \pi = 3.14\ldots$
- $R_{S_2}(\varphi) = \dfrac{\varphi}{\sin \varphi},\ \varphi \in [0, \pi/2]$
- $c_{S_2} = R_{S_2}(\pi/2) = \pi/2 = 1.5708\ldots$
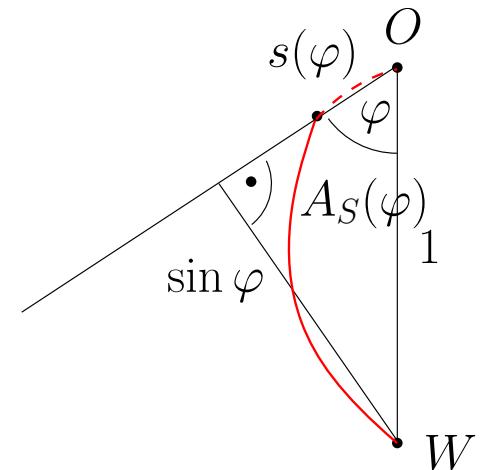- Upper bound: 1.5708, Lower Bound: 1.1547

Strategy: Design

- Reasonable strategy: Window Shopper, Street Searching, Equality approach

1. Arrive at O with angle $\gamma \leq \pi/2$
2. $R_{S_2}(\varphi) = \frac{A_S(\varphi)}{\sin \varphi} = c_{\text{opt}}$ forall $\varphi \in [0, \gamma]$

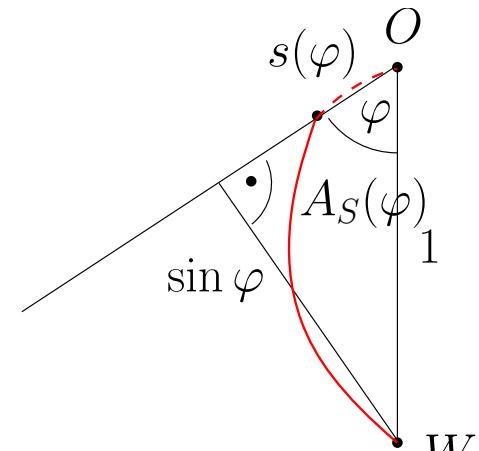- Design by requirements
- Proof of optimality

Strategy: Design

- $R_S(\varphi) = c_{\text{opt}}$ forall $\varphi \in [0, \gamma]$

-

$$
\begin{aligned}
c_{\text{opt}} &= \frac{A_S(\varphi)}{\sin \varphi} = \frac{\int_0^\varphi \sqrt{s'(t)^2 + s(t)^2}\,dt}{\sin \varphi} \Leftrightarrow \\
c \cdot \cos \varphi &= (c \sin \varphi)' = \sqrt{s'(\varphi)^2 + s(\varphi)^2}
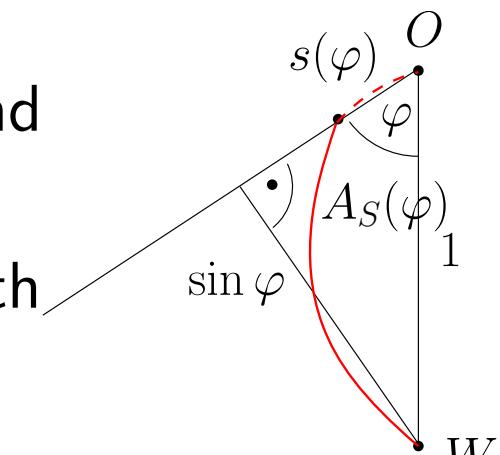\end{aligned}
$$

- $s(0) = 1$, $s(\gamma) = 0$, $s(\varphi) > 0$ for $\varphi \in (0, \gamma)$
- Solution: $s'(\varphi) = \pm\sqrt{c_{\text{opt}}^2 \cdot \cos^2 \varphi - s^2(\varphi)}$
- $s'(\varphi) = -\sqrt{c_{\text{opt}}^2 \cdot \cos^2 \varphi - s^2(\varphi)}$, decreasing radius!
- Trick: $s(\varphi) := c_{\text{opt}} r(\varphi)$ then
  differential equation: $r'(\varphi) = -\sqrt{\cos^2 \varphi - r^2(\varphi)}$
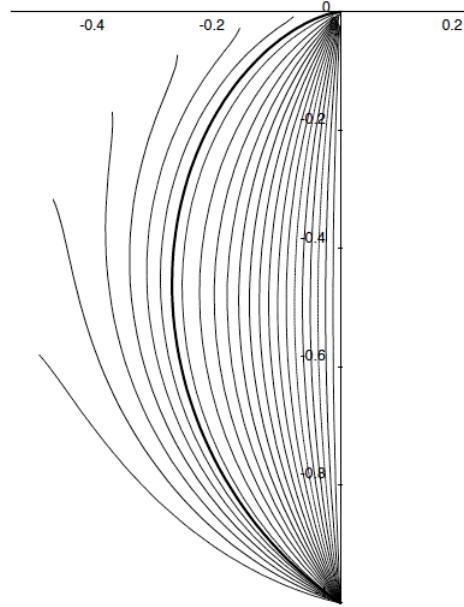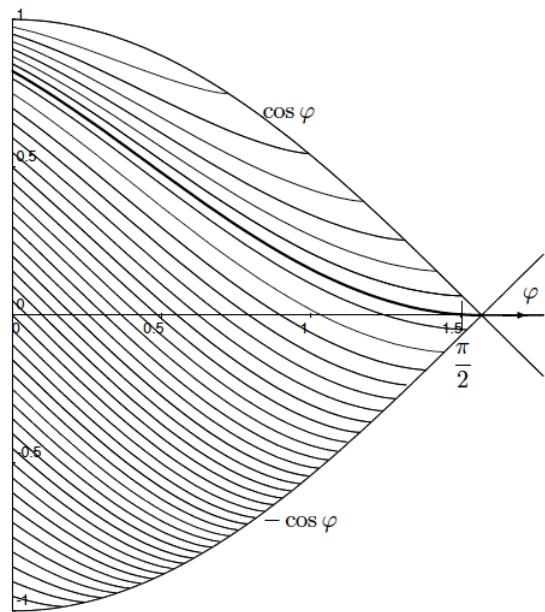
Strategy: Design

- $s(\varphi) := c \cdot r(\varphi)$

- $s(0) = 1$, $s(\gamma) = 0$, $s(\varphi) > 0$ for $\varphi \in (0, \gamma)$

- Ordinary differential equation:
  (1) $r'(\varphi) = -\sqrt{\cos^2 \varphi - r^2(\varphi)}$

- $r(0) = \frac{1}{c}$, $r(\gamma) = 0$, $r(\varphi) > 0$ for $\varphi \in (0, \gamma)$ and $r(\varphi) \leq \cos \varphi$

- Find minimum c such that (1) has solution with above properties

- Transformation (1) into:
  $w'(x) = (w^2(x) + 1)(1 - w(x) \cot x)$ no closed from solution!
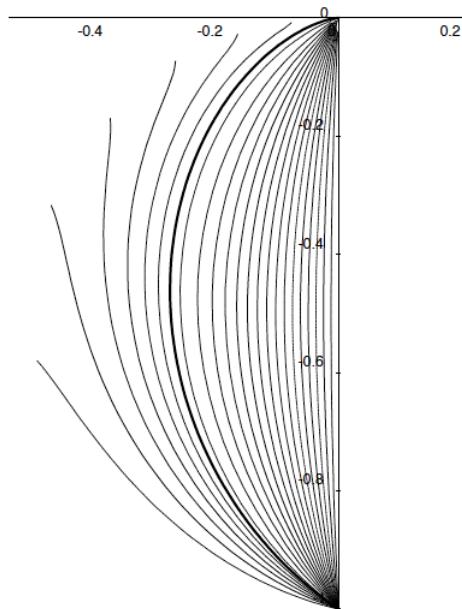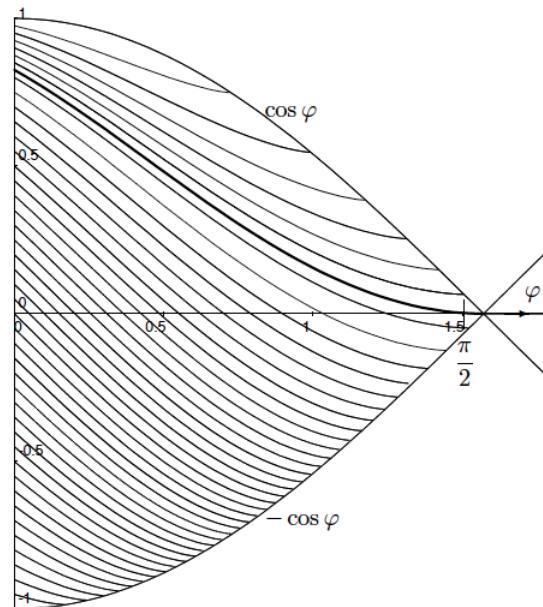
- Only numerical solutions!

Strategy: Design

Ordinary differential equation: $r'(\varphi) = -\sqrt{\cos^2 \varphi - r^2(\varphi)}$
$r(0) = \frac{1}{c}$, $r(\gamma) = 0$, $r(\varphi) > 0$ for $\varphi \in (0, \gamma)$ and $r(\varphi) \leq \cos \varphi$

Starting values $r(0) = \frac{1}{c}$, unique numerical solution! $r(\pi/2) = 0$!
Gives smallest c!

Strategy: Design

Lemma: For $r'(\varphi) = -\sqrt{\cos^2 \varphi - r^2(\varphi)}$ there is a unique solution $r(\varphi)$ satisfying $r(\pi/2) = 0$ and $r(\varphi) > 0$ for $\varphi \in [0, \pi/2)$.

Starting values $r(0) = \frac{1}{c}$, unique numerical solution! $r(\pi/2) = 0$

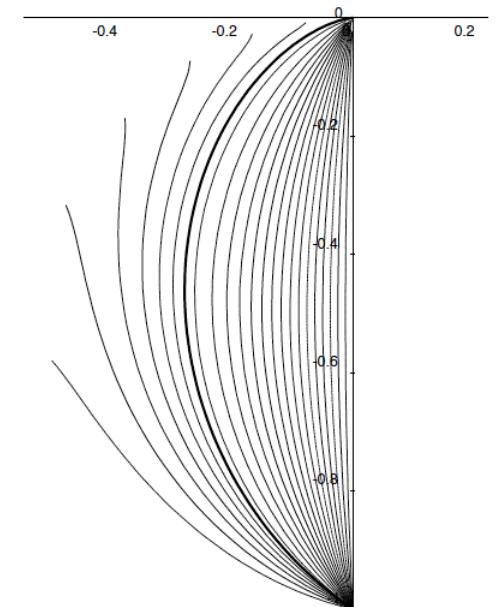Strategy $s(\varphi) = c \cdot r(\varphi)$, $c = 1.21218\ldots$

Strategy: Optimality

Theorem: The strategy S with $s(\varphi) = c \cdot r(\varphi)$ with $c = 1.21218\ldots$ is the optimal strategy for looking around a corned

- Helping Lemma: Strategy S is convex!

  Curvature formula: $\kappa := \dfrac{s^2 + 2(s')^2 - ss''}{((s')^2 + s^2)^{\frac{3}{2}}} > 0$

- Let T be arbitrary corner strategy. $s'_T(0) < \infty$
- Case I: $s'_T(0) \le s'(0) < 0$
- $s'(0) = -\sqrt{c^2 \cos(0) - s(0)^2} = -\sqrt{c^2 - 1}$
- $\Rightarrow c_T \ge \sqrt{s'_T(0)^2 + 1} \ge \sqrt{s'(0)^2 + 1} = c$
- $\Rightarrow$ T worse than S

Strategy: Optimality

Theorem: The strategy S with $s(\varphi) = c \cdot r(\varphi)$ with $c = 1.21218\ldots$ is the optimal strategy for looking around a corned

- Case II: $s_T'(0) > s'(0)$

- $\Rightarrow \exists \psi$ such that $s_T(\varphi) > s(\varphi) \; \forall \varphi \in [0, \psi]$

- Subcase I: $s_T(\psi) = s_T(\psi)$ for $\psi \leq \pi/2$:

$$c_S := \frac{A_S(\psi)}{\sin \psi} < \frac{A_T(\psi)}{\sin \psi} \leq c_T$$

- Subcase II: $s_T(\varphi) > s_T(\psi)$ for $\varphi \in (0, \pi/2]$, $\psi > \pi/2$

$$c_S := \frac{A_S(\pi/2)}{\sin \pi/2} < \frac{A_T(\pi/2) + s_T(\pi/2)}{\sin \pi/2} \leq c_T$$

- $\Rightarrow$ T worse than S