Online Motion Planning MA-INF 1314

Summersemester 17
Escape Paths/Alternative Measure

Elmar Langetepe

University of Bonn

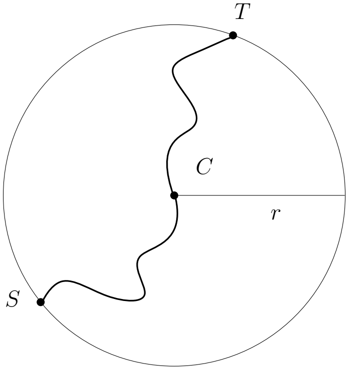
Juli 11th, 2017

Rep.: Simple escape path examples

Obviously: The diameter of any region R is always an escape path!

Theorem: The shortest escape path for a circle of radius r is a line segment of length 2r.

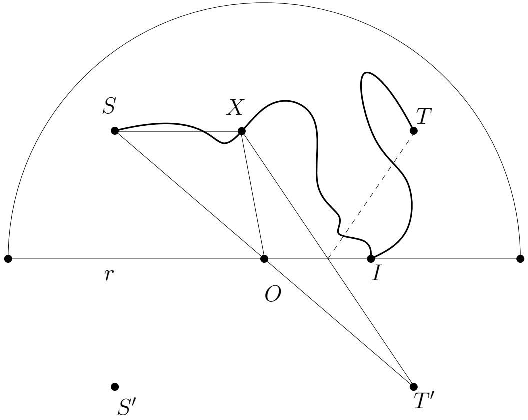
Proof: Assume there is a better escape path! Contradiction!



Rep.: Also for semicircles

Theorem: The shortest escape path for a semicircle of radius *r* is a line segment of length 2r.

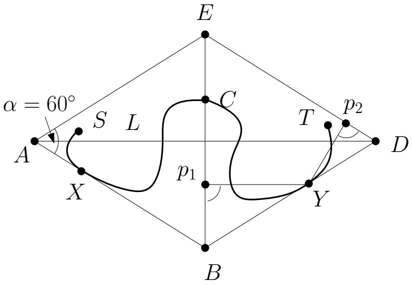
Proof: Assume there is a better escape path! Contradiction!



Rep.: More generally for a rhombus with angle 60°

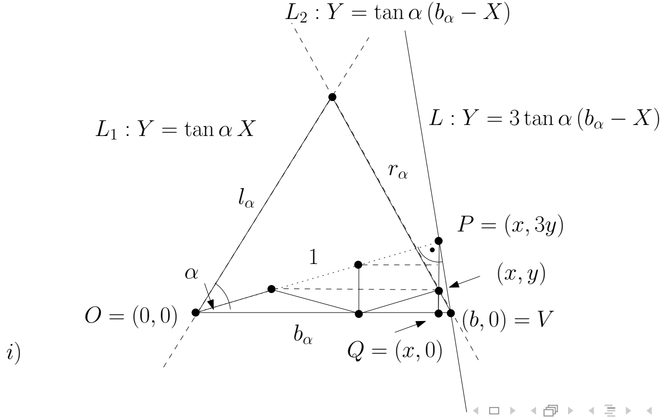
Theorem: The shortest escape path for a rhombus of diameter L with angle $\alpha = 60^\circ$ is a line segment of length L.

Proof: Assume there is a better escape path! Contradiction!



Rep.: Equilateral triangle, Zig-Zag path

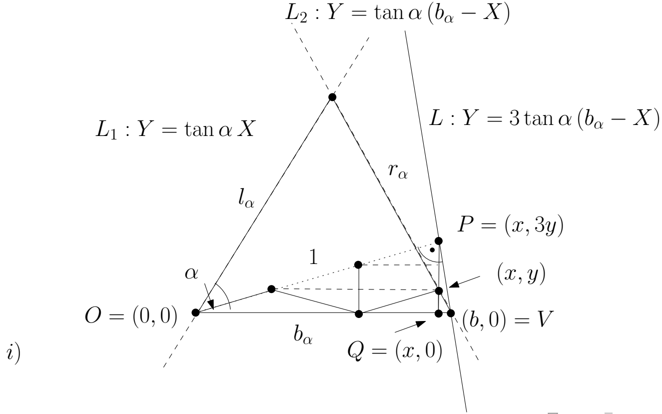
- Equilateral triangle: Besicovitch
- Zig-Zag escape path with length ≈ 0.9812
- More generally from Coulton and Movshovich (2006)
- Isosceles triangle for α and b_α
- b_α is diameter!



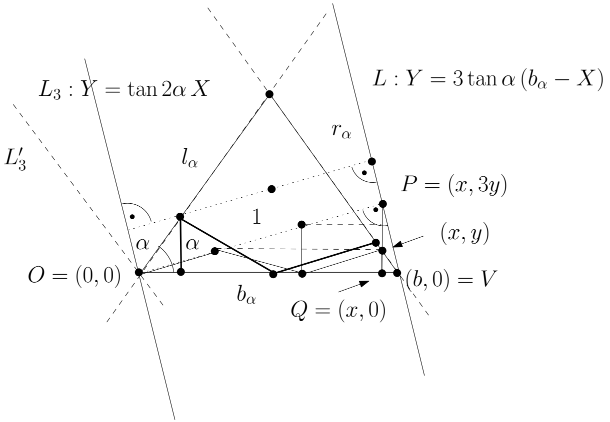
Rep.: Calculations!

Finally we determine b_α:
$y = \tan\alpha\left(b_\alpha - \frac{1}{b_\alpha}\right)$ and $x = \frac{1}{b_\alpha}$ and $x^2 + (3y)^2 = 1$ which gives

$$b_\alpha = \sqrt{1 + \frac{1}{9\tan^2\alpha}}\,.$$



There should be no better Zig-Zag path for T_α!
Line $L_3 : Y = \tan(2\alpha)$ runs in parallel with L_2. This means $-3\tan\alpha = \tan 2\alpha$ or $\tan\alpha = \sqrt{\frac{5}{3}}$.



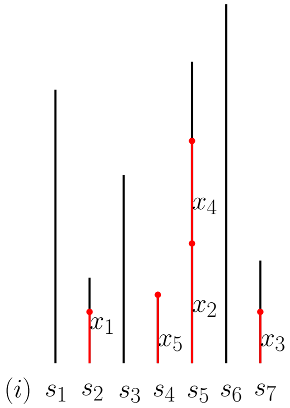
Rep.: Besicovitch triangles

Theorem: For any $\alpha \in [\arctan(\sqrt{\frac{5}{3}}), 60°]$ there is a symmetric Zig-Zag path of lenght 1 that is an escape path of T_α smaller than the diameter b_α.

- $b_\alpha = \sqrt{1 + \frac{1}{9 \tan^2 \alpha}}$
- $\alpha = 60°$: $b_\alpha = \sqrt{\frac{28}{27}}$
- $b_\alpha := 1 \Longrightarrow \sqrt{\frac{27}{28}} < 1$ is Zig-Zag path length
- Optimality? Yes!

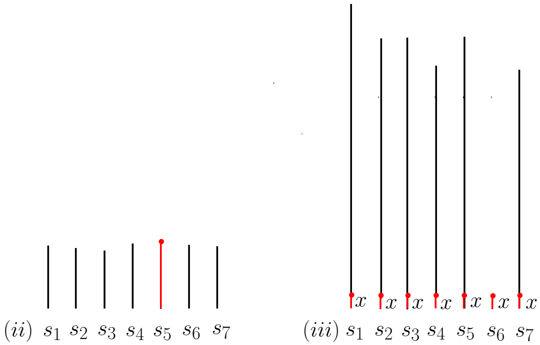
Different performance measures

- Set L_m of m line segments s_i of unknown length $|s_i|$
- Dark corridors, escape, digging for oil
- Test corridors successively
- s_{j_1} up to a certain distance x_1, then s_{j_2} for another distance x_2 and so on



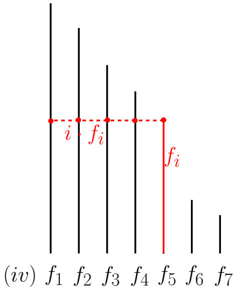
More information

- Assume distribution is known!
- $f_1 \geq f_2 \geq \cdots \geq f_m$ order of the length given
- Extreme cases! Good strategies!



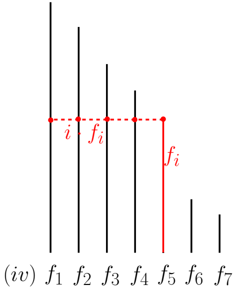
More information

- $f_1 \geq f_2 \geq \cdots \geq f_m$ order of the length given
- Check i arbitrary segments with length f_i:
  $\min_i i \cdot f_i$ is the best strategy



Known length in general

- $f_1 \geq f_2 \geq \cdots \geq f_m$ order of the length given
- Check i arbitrary segments with length f_i: $\min_i i \cdot f_i$ is a reasonable strategy
- $C(F_m, A)$ travel cost for algorithm A
- $\text{maxTrav}(F_m) := \min_A C(F_m, A)$

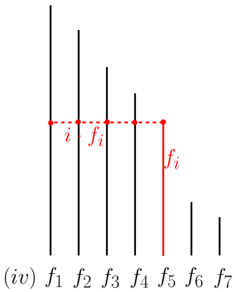


Optimal strategy for this case

Theorem: For a set of sorted distances F_m (i.e. $f_1 \geq f_2 \geq \cdots \geq f_m$) we have

$$\text{maxTrav}(F_m) := \min_i i \cdot f_i \,.$$

Proof:



Optimal strategy for this case

Theorem: For a set of sorted distances F_m (i.e. $f_1 \geq f_2 \geq \cdots \geq f_m$) we have

$$\text{maxTrav}(F_m) := \min_i i \cdot f_i \,.$$

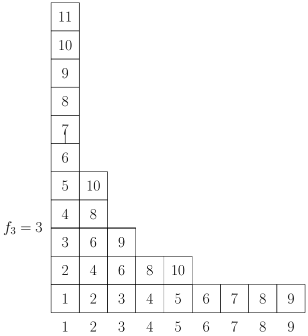
Proof:

- Arbitrary strategy A
- Less than $\min_i i \cdot f_i$ means less than $j \cdot f_j$ for any j
- Visiting depth $d_1 \geq d_2 \geq \cdots \geq d_m$
- Not reached f_1 by d_1,
  not reached f_2, since $d_1 + d_2 < 2f_2$ and $d_2 \leq d_1$ and so on
- Not successful!
- $\min_i i \cdot f_i$ always sufficient!

Online Strategy

- F_m with $f_1 \geq f_2 \geq \cdots \geq f_m$ not known
- Compete against $\text{maxTrav}(F_m) := \min_i i \cdot f_i$
- Dovetailing strategy: Rounds $c = 1, 2, 3, 4, \ldots$
- For any round c from left to right:
  Path length of segment i is *extended* up to distance $\left\lfloor \frac{c}{i} \right\rfloor$

- Dovetailing strategy: Rounds $c = 1, 2, 3, 4, \ldots$
- For any round c from left to right:
  Path length of segment i is *extended* up to distance $\left\lfloor \frac{c}{i} \right\rfloor$



Online Strategy!

Theorem: Hyperbolic traversal algorithm solves the multi-segment escape problem for any list F_m with maximum traversal cost bounded by

$$D \cdot (\mathrm{maxTrav}(F_m) \ln(\min(m, \mathrm{maxTrav}(F_m)))$$

for some constant D.

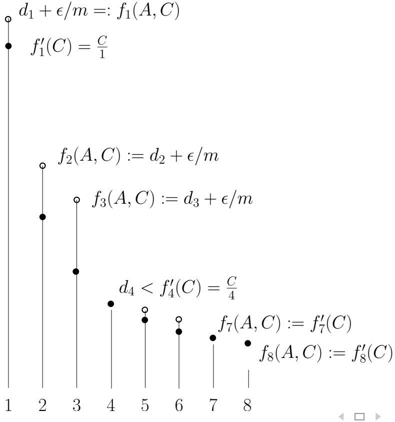
Proof:(W.l.o.g. F_m integers)

- Let $\min_i i \cdot f_i = j \cdot f_j$ for some j
- c with $c = j \cdot f_j$ exists (Round c)
- Overoll cost:

$$\sum_{t=1}^{m} \left\lfloor \frac{c}{t} \right\rfloor \leq \sum_{t=1}^{\min(m,c)} \frac{c}{t} \leq c + \int_{1}^{\min(m,c)} \frac{c}{t}\, dt = c(1 + \ln \min(m, c))\,.$$

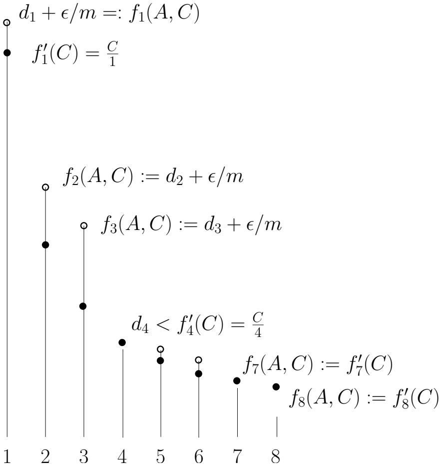
Matches Lower bound!

Theorem: For any deterministic online strategy A that solves the multi-segment escape problem we can construct input sequences $F_m(A, C)$ so that A has cost at least $d \cdot C \ln \min(C, m)$ and $\mathrm{maxTrav}(F_m(C, A)) \leq C$ holds for some constant d and arbitrarily large values C.



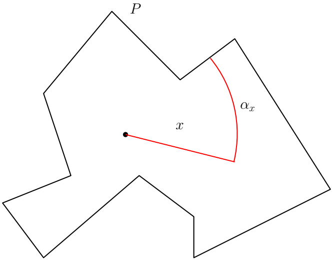
Matches Lower bound! Proof!

- C is given! $f_i'(C) = \frac{C}{i}$ (not yet fixed)
- Wait until cost $\sum_{i=1}^{m} d_i \geq d \cdot C \ln \min(C, m)$ for some d
- Fix the scenario as shown below!



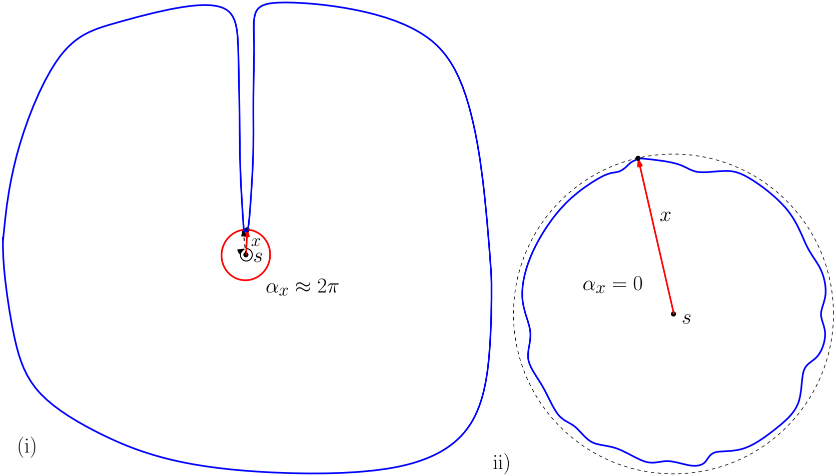
Different performance measure: Simple Polygon

- Simple polygon, escape path unknown
- Searching for different cost measure
- Polygonal extension of the list search problem
- Distance to the boundary x (estimation, given)
- Simple circular strategy $x(1 + \alpha_x)$



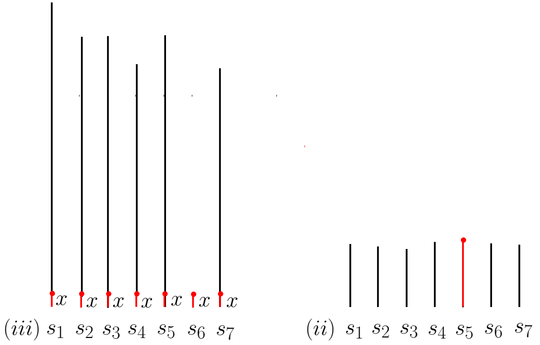
Extreme cases: Circular strategy

- Circular escape path: Distribution of the length is known
- Extreme situations: $x_1(1+2\pi)$, $x_2(1+0)$



Discrete Version! Extreme Cases!

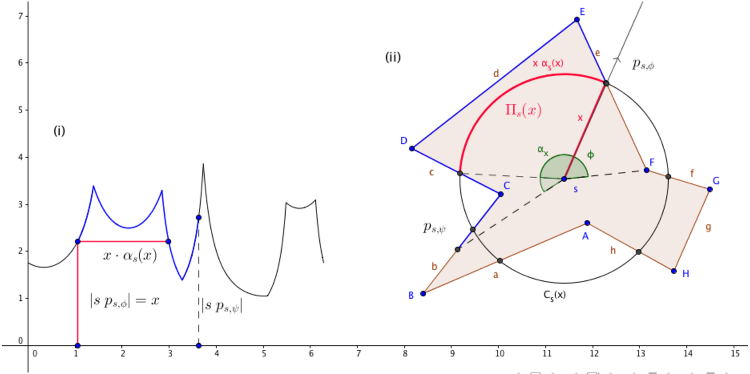
- Assume distribution is known!
- $f_1 \geq f_2 \geq \cdots \geq f_m$ order of the length given
- Extreme cases! $x_1(m)$, $x_2(1)$



- Optimal circular espape path for $s \in P$: $\Pi_s(x)$
- For any distance x a worst-case $\alpha_s(x)$
- In total: $\min_x \; x(1+\alpha_s(x))$

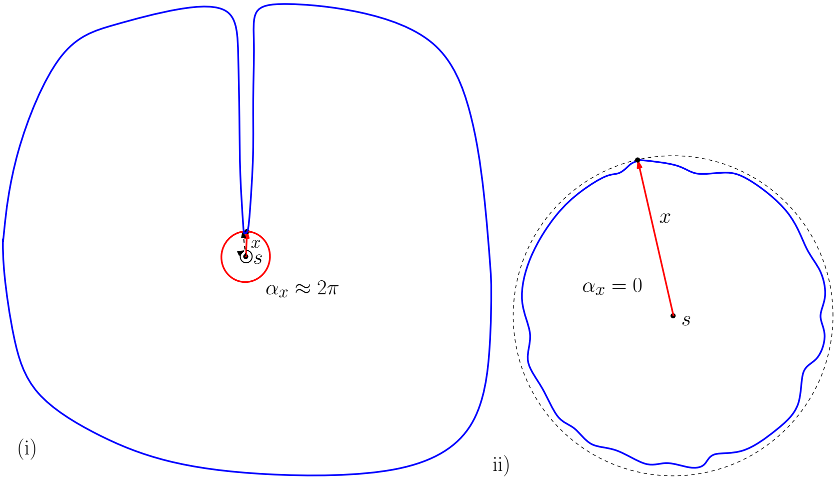
$$\Pi_s := \min_x \Pi_s(x) = \min_x x(1+\alpha_s(x)) \; .$$

- Radial dist. function interpretation: Area plus height!



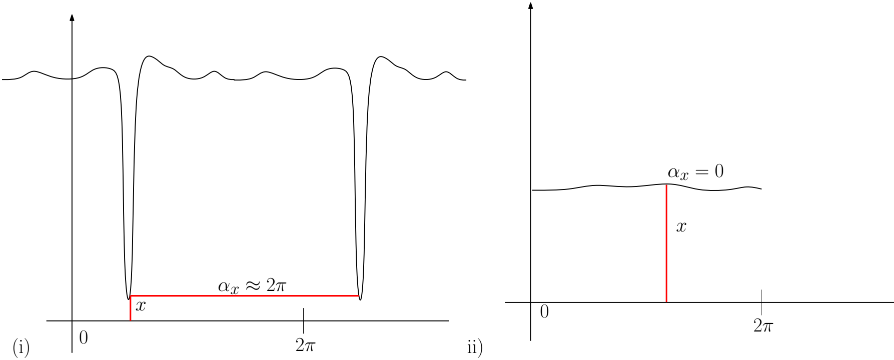
Extreme cases: Radial dist. function

- Circular escape path: Distribution of the length is known
- Extreme situations: $x_1(1 + 2\pi)$, $x_2(1 + 0)$



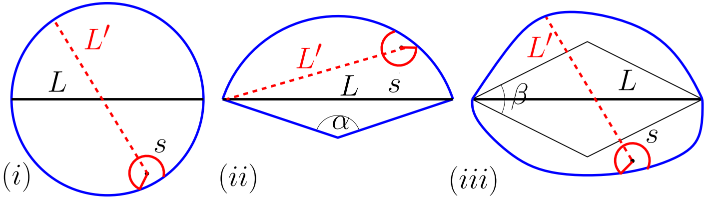
Radial distance function of extreme cases

- Optimal circular espape path
- Hit the boundary by 90 degree wedge
- Area plus height! $\min_x \; x(1 + \alpha_x)$



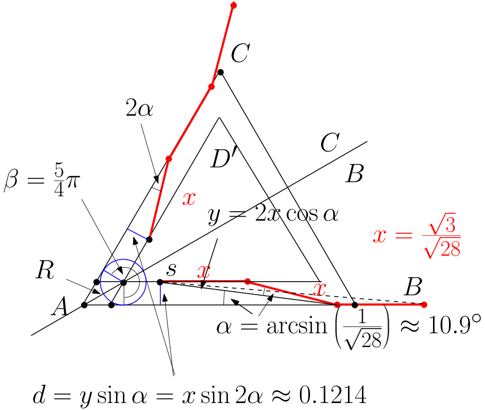
Different justifications

- Simple, computation (polynomial), star-shaped vs. convex
- Natural extension of the discrete certificate (Kirkpatrik)
- Outperforms escape paths for known cases (diameter)



Outperforms Zig-Zag path

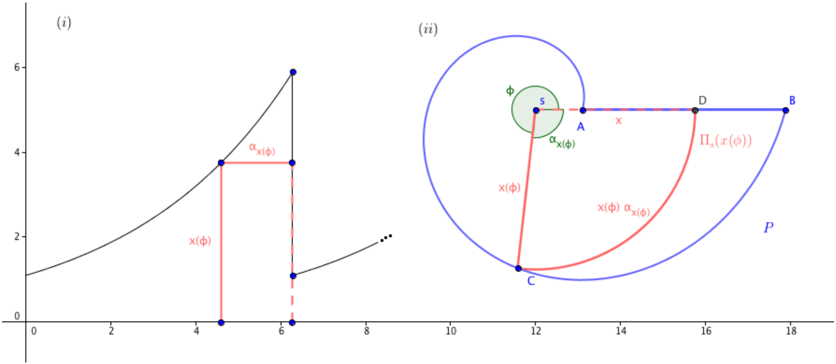
- For any position, better than the Zig-Zag path
- Formal arguments!
- Zig-Zag cannot end in farthest vertex: Region *R*!



$0.125 \times (5\pi/4 + 1) < 2x = 2\frac{\sqrt{3}}{\sqrt{28}}$

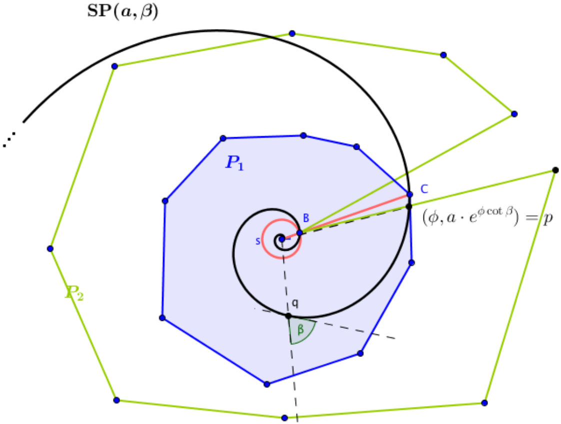
Interesting example

- Distance distribution exactly resembles the polygon
- Analogy to discrete case! Sorting!
- Log. spiral α_x for any x is known:
  $x(\phi) \cdot (1 + \alpha_{x(\phi)})$ with $\alpha_{x(\phi)} = 2\pi - \phi$ and $x(\phi) = A \cdot e^{\phi \cot \beta}$



Online Approximation!

- Inside a polygon P at point s, totally unknown
- Leave the polygon, compare to certificate path for $s \in P$
- Dovetailing strategy (discr. case)! Now spiral strategy (a, β)!



Analysis of a spiral strategy!

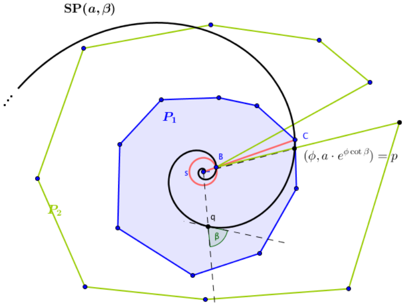
- Assume certificate: $x(1+\alpha_x)$ for s
- Spiral reach distance $x = a \cdot e^{(\phi-\alpha_x)\cot(\beta)}$ at angle ϕ
- Worst-case success at angle ϕ! (Increasing for α_x distances!)
- Ratio:

$$f(\gamma, a, \beta) = \frac{\frac{a}{\cos\beta} \cdot e^{\phi\cot\beta}}{a \cdot e^{(\phi-\gamma)\cot\beta}(1+\gamma)} = \frac{e^{\gamma\cot\beta}}{\cos\beta(1+\gamma)} \text{ for } \gamma \in [0, 2\pi]$$

- γ represents possible α_x!
- (β, a) represents the spiral strategy!
- Independent from a!
- How to choose β?

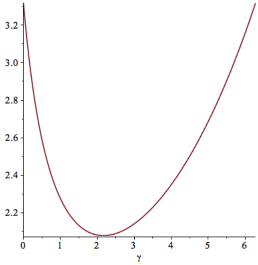
How to choose β?

- Ratio: $f(\gamma, \beta) = \frac{e^{\gamma \cot \beta}}{\cos \beta (1+\gamma)}$ for $\gamma \in [0, 2\pi]$
- Balance: Choose β s.th. extreme cases have the same ratio
- $f(0, \beta) = \frac{1}{\cos \beta} = \frac{e^{2\pi \cot \beta}}{\cos \beta (1+2\pi)} = f(2\pi, \beta)$
- $\beta = \text{arccot}\left(\frac{\ln(2\pi+1)}{2\pi}\right) = 1.264714\ldots$



Balance the extreme cases!

- $\beta := \text{arccot}\left(\frac{\ln(2\pi+1)}{2\pi}\right) = 1.264714\ldots$
- Ratio: $f(\gamma, \beta) = \frac{e^{\gamma \cot \beta}}{\cos \beta (1+\gamma)}$ for $\gamma \in [0, 2\pi]$
- $f(0, \beta) = f(2\pi, \beta) = 3.31864\ldots$
  and $f(\gamma, \beta) < 3.31864\ldots$ for $\gamma \in (0, 2\pi)$



Spiral strategy for $\beta = 1.264714\ldots$

Theorem: There is a spiral strategy for any unknown starting point s in any unknown environment P that approximates the certificate for s and P within a ratio of 3.31864.