Online Motion Planning MA-INF 1314
Online TSP, Shortcut algorithm

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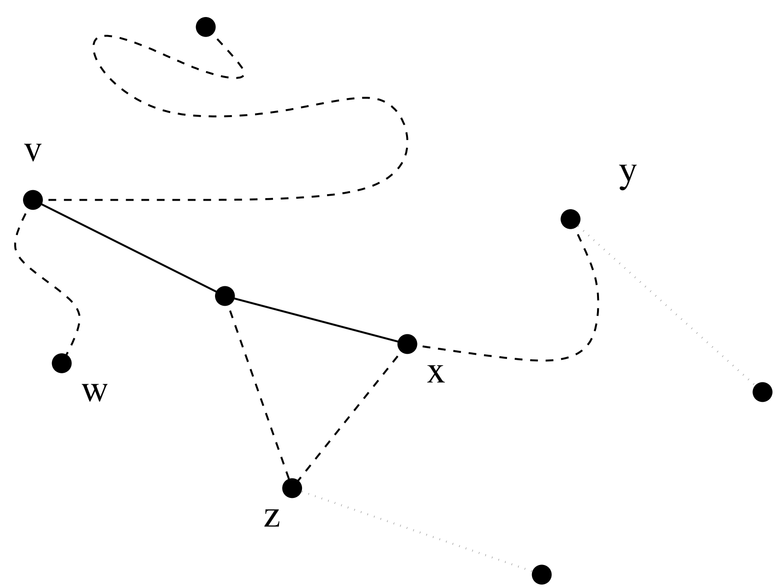
Online TSP Problem

- Unknown graph $G = (V, E)$
- Visit all vertices and return
- *Traffic sign modell*
- Planar graphs
- DFS search
- Problem of large edges

Shortcut algorithm

- Vertices: Visited, Boundary, Unknown
- Edges: Explored, Boundary, Unknown
- Shortest path $d_G(v, w)$

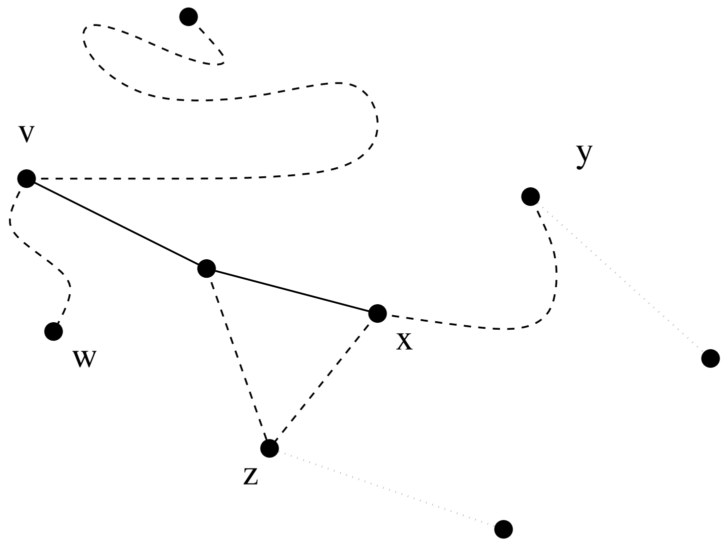


Shortcut algorithm

- Boundary edge (x, y) *blocks* boundary edge (v, w)

1. $|xy| < |vw|$ and
2. $d_G(v, x) + |xy| < (1 + \delta)|vw|$

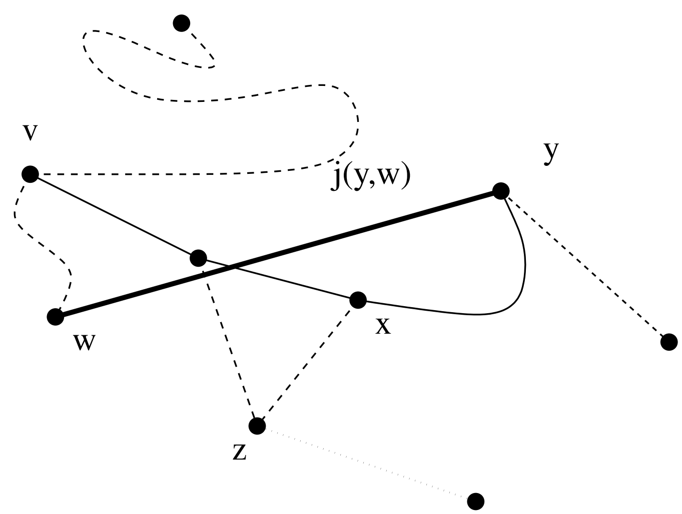
- Later: You eventually move from y over x to v!



Shortcut algorithm

- *Shortcut* non-blocked boundary edge (x, y), charge edge (x, y)
- Blocked edge (v, w) becomes unblocked!
- Artificial edge $j(y, w)$ of length $d_G(v, w) < (2 + \delta)|vw|$

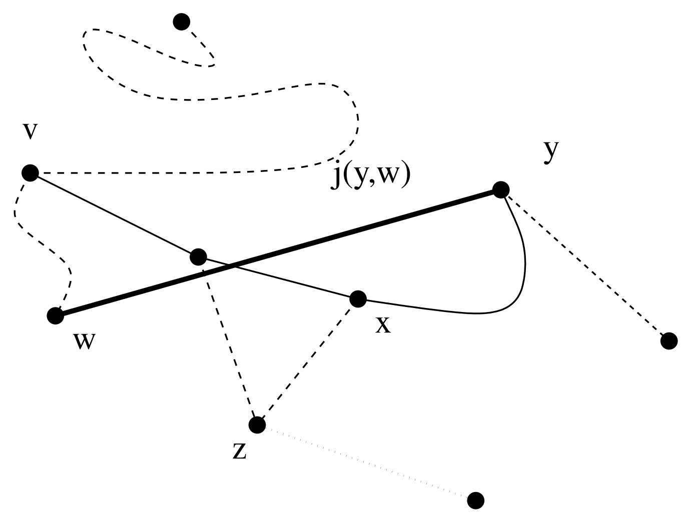
1. $|xy| < |vw|$ and
2. $d_G(v, x) + |xy| < (1 + \delta)|vw|$



Shortcut algorithm

- Use non-blocked DFS edge (x, y), charge edge (x, y)
- Use jump DFS edge $j(y, w)$, charge edge (v, w)

1. $|xy| < |vw|$ and
2. $d_G(v, x) + |xy| < (1 + \delta)|vw|$



Pseudocode: Shortcut(x, y, G)

Traveling from x, we have visited y for the first time

for all Boundary edges (v, w) **do**

  if Visiting y caused $Block((v, w))$ to become empty **then**

    Build jump edge $j(y, w)$, append jump edge

      on $Incident(y)$ and $Incident(w)$

  end if

end for

for all Edges $(y, z) \in Incident(y)$ **do**

  if z is boundary vertex and (y, z) shortcut **then**

    Traverse the egde (y, z); Shortcut(y, z, G)

  else if z is boundary vertex and (y, z) is jump edge **then**

    Move to z along the shortest known path; Shortcut(y, z, G)

  end if

end for

Return to x along the shortest known path

Shortcut algorithm: Analysis

Lemma: (structural properties)

1. Any vertex of a graph G will be visited.
2. If during execution bound. edge (x, y) blocks bound. edge (v, w), then (x, y) blocks (v, w) until either y or w is visited.
3. If after traversing a boundary edge (x, y) another boundary edge (v, w) became a shortcut and a jump edge $j(y, w)$ was added, we have $d_G(y, w) < (2 + \delta)|vw|$.

1. and 3. by construction!

2. Bound. edges until neither y nor w is visited, condition remains

Shortcut algorithm: Analysis

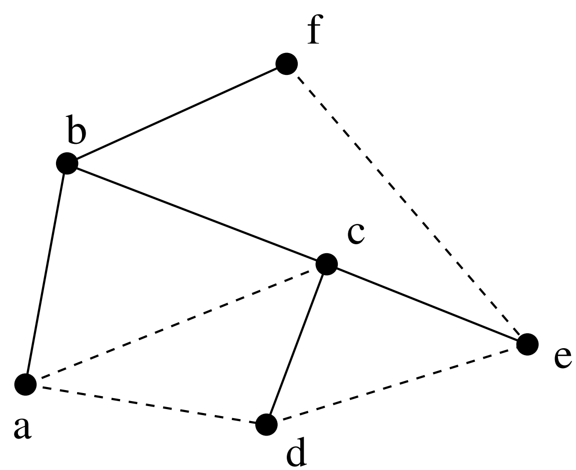
Theorem: For planar graphs the above Shortcut algorithm is 16-competitive.

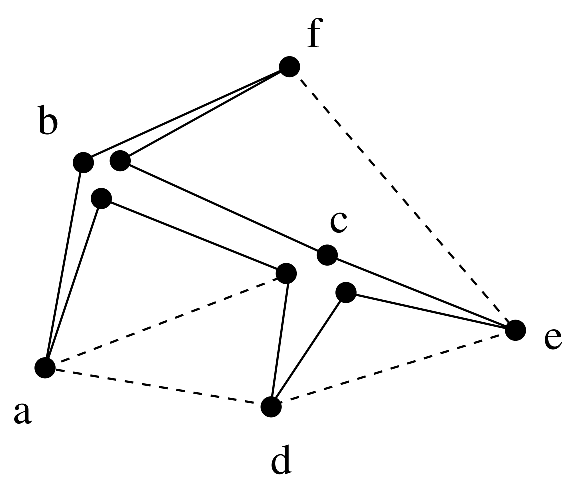
Proof:

- Charged edges! Shortcuts, jump edges, set P
- $2(2+\delta)|P|$ in total! Factor 2 DFS forth and back!
- MST of G, $|OPT| \geq |MST|$
- Show $|P| \leq (1+2/\delta)|MST|$
- $2(2+\delta)|P| \leq 2(2+\delta)(1+2/\delta)|MST| \leq 2(2+\delta)(1+2/\delta)|OPT|$
- Minimize $2(2+\delta)(1+2/\delta)$, $\delta = 2$, 16!

Shortcut algorithm: Analysis

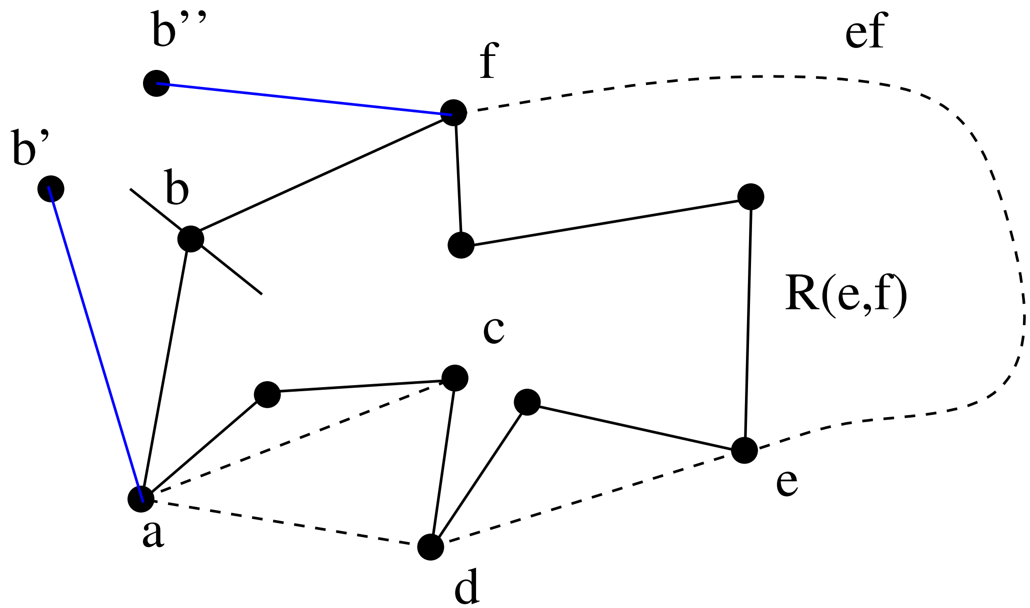
- Charged edges! Shortcuts, jump edges, set P
- Proof: $|P| \leq (1 + 2/\delta)|MST|$
- $|MST|$ with maximal number of edges form P!
- $MST \cup P$, consider ring R chord set C, $C = MST \setminus P$
- Show: $|C| \leq |R|/\delta$, $|R| = 2|MST|$





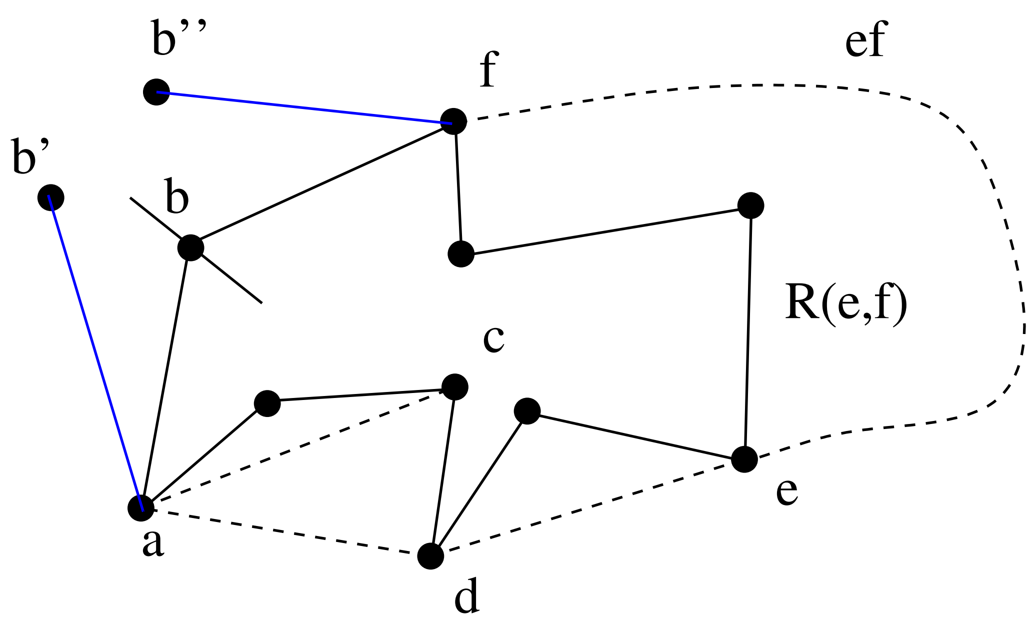
Shortcut algorithm: Analysis

- Proof: $|P| \leq (1 + 2/\delta)|MST|$
- Show: $|C| \leq |R|/\delta$, $C = MST \setminus P$
- Open the ring accordingly
- Process chords from *inside* to outside, budget $|R|$



Shortcut algorithm: Analysis

- Show: $|C| \leq |R|/\delta$, $C = MST \setminus P$
- Exchange $R(a,c)$ by (a,c), R'
- Exchange $R'(a,d)$ by (a,d), R''
- And so on from *inside* to outside



Shortcut algorithm: Analysis

- Show: $|C| \leq |R|/\delta$, $C = MST \setminus P$
- Assume: (x, y) inner chord for $R(x, y)$, Blackboard
- Show: $|R(x, y)| \geq (1 + \delta)|xy|$, *good* chords!
- Substitute $R(x, y)$ by (x, y)
- $-\delta|xy|$ from the budget of $|R|$, recursively!

Shortcut algorithm: Analysis

- Assume: (x, y) inner chord for $R(x, y)$, Blackboard
- Show: $|R(x, y)| \geq (1 + \delta)|xy|$
- By contradiction: Assume $|R(x, y)| < (1 + \delta)|xy|$
- Edge $(v, w) \in R(x, y)$ with $|vw| \geq |xy|$ is called *large*
- Show 1: There is always a large edge!
- Consider point in time t where (x, y) is charged! Must exist!
- There is a first boundary edge (v, w) in $R(x, y)$
- $d_G(x, v) + |vw| < (1 + \delta)|xy|$ because $|R(x, y)| < (1 + \delta)|xy|$ was assumed
- $|xy| > |vw|$, then (x, y) was still blocked by (v, w), contradiction!
- Thus $|xy| \geq |vw|$ holds, (v, w) is large

Shortcut algorithm: Analysis

- By contradiction: Assume $|R(x,y)| < (1+\delta)|xy|$
- Edge $(v,w) \in R(x,y)$ with $|vw| \geq |xy|$ is called *large*
- Show 2: There is large edge in $R(x,y)$ not charged!
- $(v,w) \in R(x,y)$ charged last point t in time
- Boundary edge on path $R(x,y) + (x,y) - (v,w)$ from v to w at t
- (a,b) first such while moving from v to w
- $|R(x,y)| < (1+\delta)|xy|$ and $|xy| \leq |vw|$
- Also $d_G(v,a) + |ab| \leq |R(x,y)| + |xy| - |vw|$
- $d_G(v,a) + |ab| < (1+\delta)|vw|$, (v,w) not blocked
- $|ab| \geq |vw|$, and (a,b) is large!

Shortcut algorithm: Analysis

- By contradiction: Assume $|R(x, y)| < (1 + \delta)|xy|$
- Show 2: There is always a large edge (a, b) not charged!
- $MST - (a, b) + (x, y)$ has smaller cost than MST for $|ab| > |xy|$
- For $|ab| = |xy|$ the spanning tree $MST - (a, b) + (x, y)$ induces less chords
- Contradiction! $|R(x, y)| \geq (1 + \delta)|xy|$

Shortcut algorithm: Analysis

- Show: $|C| \leq |R|/\delta$, $C = MST \setminus P$
- Assume: (x, y) inner chord for $R(x, y)$, Blackboard
- Show: $|R(x, y)| \geq (1 + \delta)|xy|$, *good* chords!
- Exchange $R(x, y)$ by (x, y), R'
- Cost $-\delta|xy|$ from the budget $|R|$, recursively!
- Next inner chord (x', y'): $|R'(x', y')| \geq (1 + \delta)|x'y'|$
- $|R| - \delta|xy| - \delta|x'y'|$ and so on
- $|R| - \delta|C| \geq 0$

Shortcut algorithm: Analysis

Theorem: For planar graphs the above Shortcut algorithm is 16-competitive.

Proof:

- P set of charged edges
- $2(2+\delta)|P|$ in total! Factor 2 DFS forth and back!
- MST of G, $|OPT| \geq |MST|$
- $|P| \leq (1+2/\delta)|MST|$
- By $|C| \leq |R|/\delta$, $|R| = 2|MST|$, $C = MST \setminus P$
- $2(2+\delta)|P| \leq 2(2+\delta)(1+2/\delta)|MST| \leq 2(2+\delta)(1+2/\delta)|OPT|$
- Minimize $2(2+\delta)(1+2/\delta)$, $\delta = 2$, 16!

Shortcut algorithm: Computational cost

Lemma: The computational cost of the Shortcut algorithm is bounded by $O(n^2 \log n)$ for a planar graph of n vertices.

- Lists $BlockedBy(e)$ and $Block(e)$, cross pointers
- Detect boundary edge (x, y): Init. $BlockedBy((x, y))$, $Block((x, y))$
- Single source Dijkstra for x
- All (v, w) with $|xy| < |vw|$ and $d_G(v, x) + |xy| < (1 + \delta)|vw|$
- All (a, b) with $|ab| < |xy|$ and $d_G(x, a) + |ab| < (1 + \delta)|xy|$
- $O(n^2 \log n)$, planar, $O(|E|) = O(|V|) = n$
- Update: $BlockedBy(e)$ and $Block(e)$
- Traverse jump edges: Again Dijkstra! $O(n)$ jump edges!$O(n^2 \log n)$

- Explored (x, y) no longer blocks: for any $(v, w) \in BlockedBy((x, y))$ remove (x, y) in $Block((v, w))$, $O(1)$
- $O(n)$ lists in total
- Any edge inserted, deleted once! $O(n^2)$!