5. Properties of Abstract Voronoi Diagrams

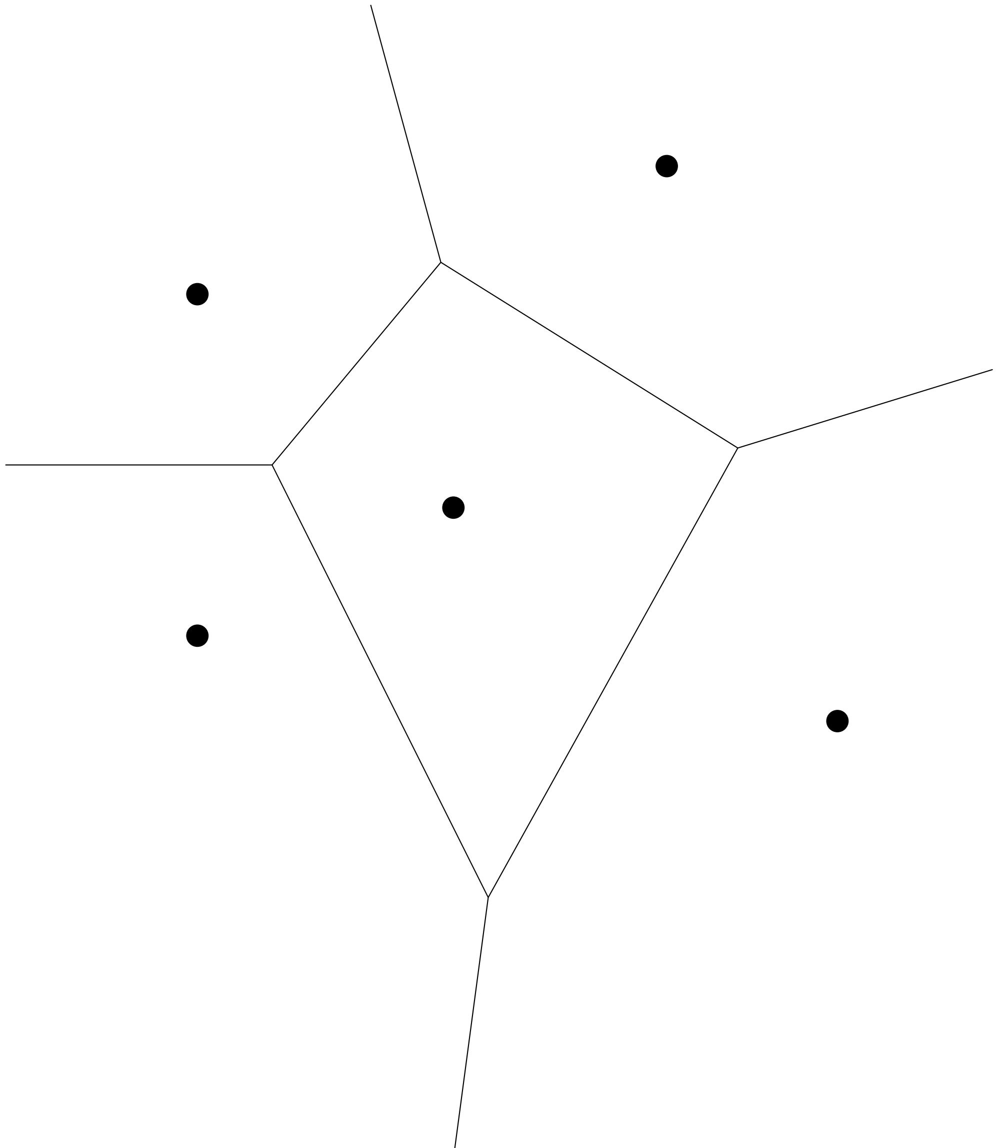
5.1 Euclidean Voronoi Diagrams

Voronoi Diagram: Given a set S of n point sites in the plane, the Voronoi diagram $V(S)$ of S is a planar subdivision such that

- Each site $p \in S$ is assigned a Voronoi region denoted by $\text{VR}(p, S)$
- All points in $\text{VR}(p, S)$ share the same nearest site p in S

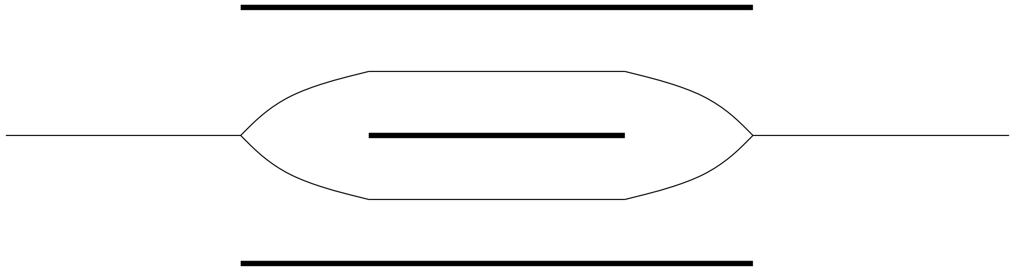
Voronoi Edge: The common boundary between two adjacent Voronoi regions, $\text{VR}(p, S)$ and $\text{VR}(q, S)$, i.e., $\text{VR}(p, S) \cap \text{VR}(q, S)$,is called a *Voronoi edge*.

Voronoi Vertex: The common vertex among more than two Voronoi regions is called a *Voronoi vertex*.

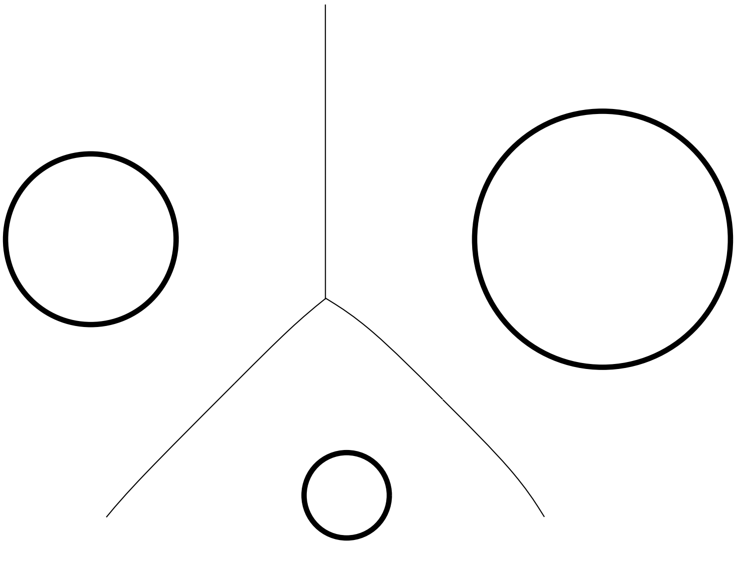


The Euclidean Voronoi diagram can be computed in $O(n \log n)$ time

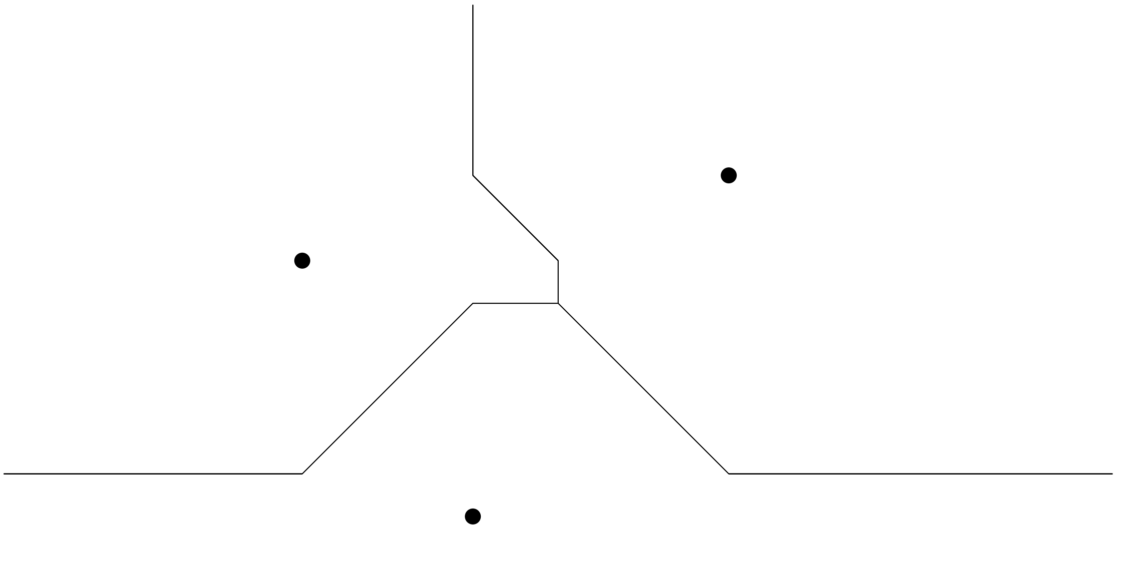
- Line Segment Voronoi Diagram



- Circle Voronoi Diagram



- Voronoi Diagram in the L_1 metric



5.2 Bisecting Systems

- For two sites $p, q \in S$, the bisector $\boldsymbol{J(p,q)}$ between p and q is defined as $\{x \in R^2 \mid d(x,p) = d(x,q)$

- $J(p,q)$ partitions the plane into two half-planes
  - $D(p,q) = \{x \in R^2 \mid d(x,p) < d(x,q)\}$
  - $D(q,p) = \{x \in R^2 \mid d(x,q) < d(x,p)\}$

- $\mathrm{VR}(p,S) = \bigcap_{q \in S \setminus \{p\}} D(p,q)$

- $V(p,S) = R^2 \setminus \bigcup_{p \in S} \mathrm{VR}(p,S)$
  - consists of Voronoi edges.

5.3 Abstract Voronoi Diagrams

A unifying approach to computing Voronoi diagrams among different geometric sites under different distance measures.

A bisecting system $\mathcal{J} = \{J(p,q) \mid p, q \in S\}$ for a set S of sites (indices)

A bisecting system $\mathcal{J}$ is **admissible** if $\mathcal{J}$ satisfies the following axioms

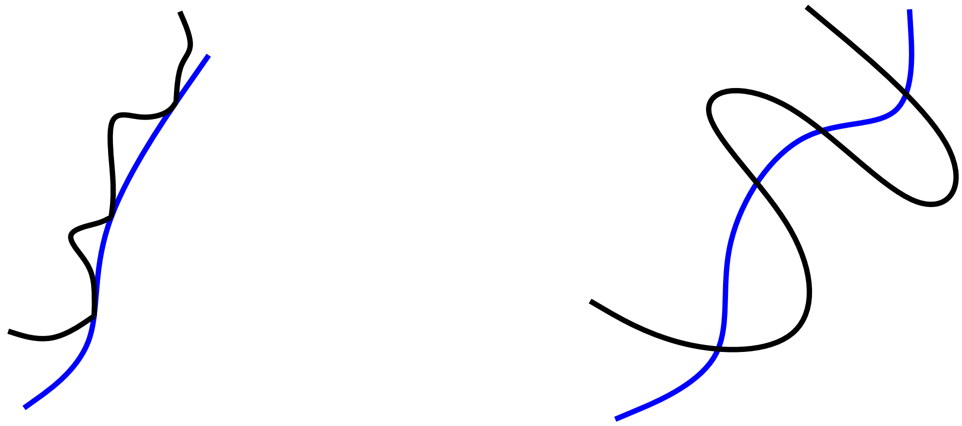
(A1) Each bisecting curve in $\mathcal{J}$ is homeomorphic to a line (not closed)

(A2) For each non-empty subset S' of S and for each $p \in S'$, $\mathrm{VR}(p,S')$ is path-connected.

(A3) For each non-empty subset S', $R^2 = \bigcup_{p \in S'} \overline{\mathrm{VR}(p,S')}$

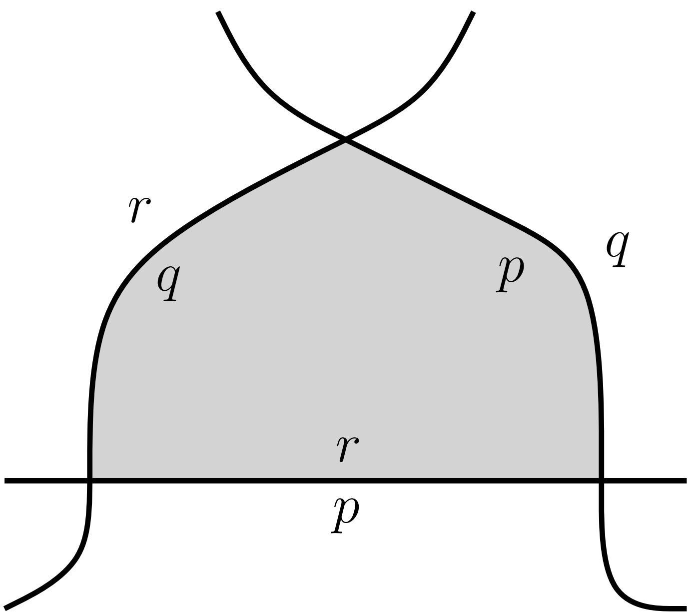
(A4) Any two curves in $\mathcal{J}$ have only finitely many intersection points, and these intersections are transversal.

- (A1) can be written as "Each curve in $\mathcal{J}$ is unbounded. After stereographic projection to the sphere, it can be completed to a closed Jordan curve through the north pole."
- (A4) can be removed through several complicated proofs.

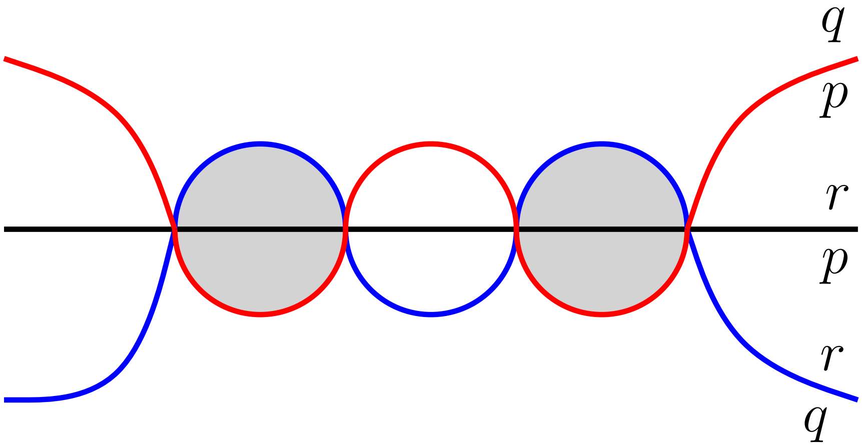


Not Traversal Traversal

Not Admissible

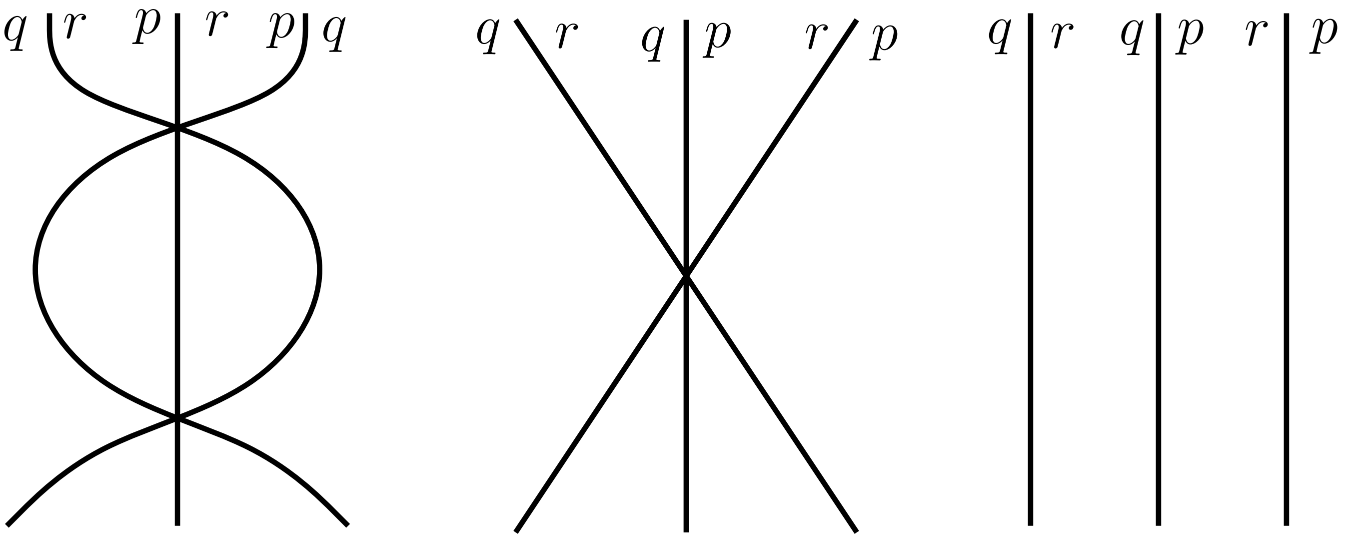


Non-Man Land



Disconnected

Three possibilities of an admissible system for three sites



Abstract Voronoi Diagrams

- A category of Voronoi diagrams
  - points in any convex distance function
  - Karlsruhe metric
  - Line segments and convex polygons of constant size

5.3 Basic Properties

Lemma 1
Let $(S, \mathcal{J})$ be a bisecting curve system. The the following assetions are equivalent.

1. If p, q, and r are pairwise different sites in S, then $D(p,q) \cap D(q,r) \subseteq D(p,r)$ (Transitivity)
2. For each nonempty subset $S' \subseteq S$, $R^2 = \bigcup_{p \in s'} \overline{\mathrm{VR}(p, S')}$

Proof:

(2) $\rightarrow$ (1)

- Let z be a point in $D(p,q) \cap D(q,r)$.
- By (2), there must be a site $t \in S' = \{p,q,r\}$ such that $z \in \mathrm{VR}(t, S')$.
- If $t = p$, $z \in \mathrm{VR}(p, S') \subseteq D(p,r)$; otherwise
  - $z \in \mathrm{VR}(q, S') \subseteq D(q,p)$, contradicting $z \in D(p,q)$
  - $z \in \mathrm{VR}(r, S') \subseteq D(r,q)$, contradicting $z \in D(q,r)$

(1) $\rightarrow$ (2)

- By induction on $|S'|$.
- If $|S'| = 2$, the assertion is immediate.
- The case where $|S'| = 3$ follows directly from (1)
- Let z be a point in the plane. By induction hypothesis, to each $p \in S'$, there exists a site $c(p) \neq p$ such that $z \in \mathrm{VR}(c(p), S' \setminus \{p\})$

  case 1: There exists $v \neq w$ such that $c(v) = c(w)$. Then
  $z \in \mathrm{VR}(c(v), S' \setminus \{v\}) \cap \mathrm{VR}(c(v), S' \setminus \{w\}$
  $\subset \mathrm{VR}(c(v), S' \setminus \{v\} \cap D(c(v), v) = \mathrm{VR}(c(v), S')$

  case 2 The mapping c is injective. Let p, v, w be scuh that $|\{p, c(p), v, w\}| = 4$. Since $c(v) \neq c(w)$, one of them is different p. We assume $c(v)$ is different from p. Since $c(v) \neq c(p)$ we obtain the contradiction:
  $z \in \mathrm{VR}(c(p), S' \setminus \{p\}) \subseteq D(c(p), c(v))$
  $z \in \mathrm{VR}(c(v), S' \setminus \{v\}) \subseteq D(c(v), c(p))$

Theorem

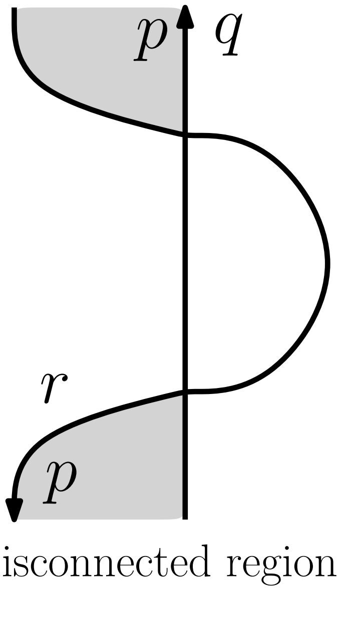
A bisecting curve system $(S, \mathcal{J})$ is *admissible* if and only if the following conditions are fulfilled.

1. $D(p,q) \cap D(q,r) \subseteq D(p,r)$ holds for any three sites p, q, r, in S
2. Any two curve $J(p,q)$ and $J(p,r)$ cross at most twice and do not constitude a clockwise cycle in the plane

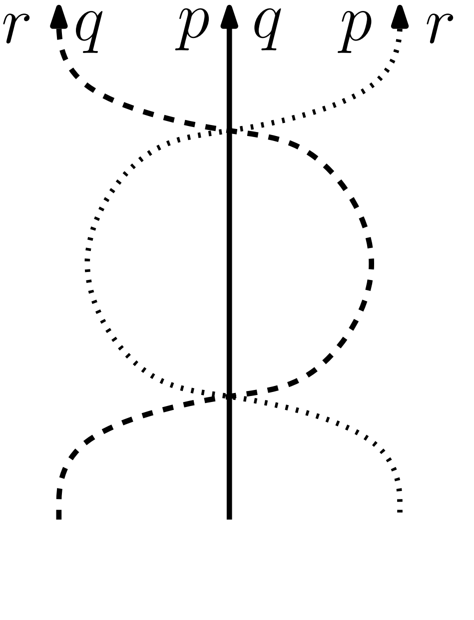
proof

$\rightarrow$

- By Lemma 1, concentrate on the connectedness of Voronoi regions.
- Consider an infinitely large bounded curve Γ which contains all intersections among curves in $\mathcal{J}$
- For any $p, q, r \in S$, $V(\{p,q,r\})$ encircled by Γ is a planar graph with exacyly 4 faces each of whose vertices is of degree at least 3.
- By the Euler Formula, the planar graph gas at most 4 vertices
- Since at least two edges of the original diagram tend to infinity, two vertices must be situated in Γ.
- $J(p,q)$ and $J(p,r)$ cross at most twice since each intersection between them is a Voronoi vertex by definition.
- A simple case analysis shows no clockwise cycle raising from $J(p,q)$ and $J(p,r)$



disconnected region



←

- The case analysis shows that for any 3-element subset S' of S, all Voronoi regions in $V(S')$ is connected.
- We prove by induction on m: If $R = \mathrm{VR}(p, \{p, q_1, q_2, \ldots, q_m\})$ is connected, then $R \cap D(p, q_{m+1}) = \mathrm{VR}(p, \{p, q_1, q_2, \ldots, q_{m+1}\})$ is connected.
- Let $J(p, q)$ be oriented such that $D(p, q)$ is on its left side.
- Assume the contrary that $R \cap D(p, q_{m+1})$ were not connected.
- If $R \cap D(p, q_{m+1})$ is bounnded, C be ∂R and $J(p, q_{m+1})$ would form a clockwise cycle.
  - For $\exists i \leq m$, $J(p, q_i)$ and $J(p, q_{m+1})$ form a clockwise cycle.
  - There exists a contradiction
- Otherwise, we intersect R with the inner domain of Γ, and C' be its contour.
  - The same reasoning applies to C' and $J(p, q_{m+1})$