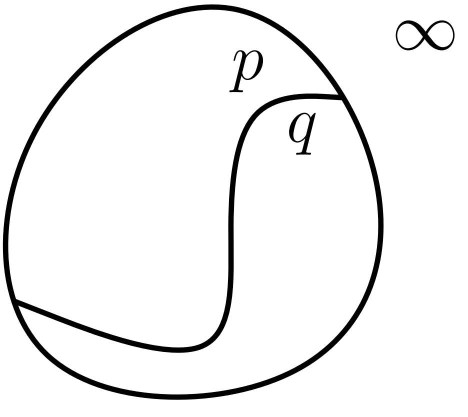
6. Cosntruction of AVD

Finite Part of AVD

- Let Γ be a simple closed curve such that all intersections between bisectring curve lie inside the inner domain of Γ
- Consider a site ∞, define $J(p, \infty) = J(\infty, p)$ to be Γ for all sites $p \in S$, and $D(\infty, p)$ to be the outer domain of Γ for all sites $p \in S$.

Incremental Construction

- Let $s_1, s_2, \ldots, s_n$ be a random squence of S
- Let R_i be $\{\infty, s_1, s_2, \ldots, s_i\}$
- Iteratively construct $V(R_2), V(R_3), \ldots, V(R_n)$



General Position Assumption

- No $J(p, q)$, $J(p, r)$ and $J(p, t)$ intersect the same point for any four distinct sites, $p, q, r, t \in S$
  $\rightarrow$ Degree of a Voronoi vertex is 3

Remark

- For $1 \leq i \leq n$ and for all sites $p \in R_i$, $\mathrm{VR}(p, R_i)$ is simply connected, i.e., path connected and no hole
- If $J(p, q)$ and $J(p, r)$ intersect at a point x, $J(q, r)$ must pass through x

Basic Operations

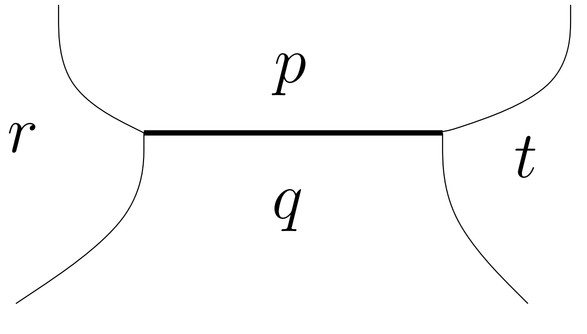
- Given $J(p,q)$ and a point v, determine $v \in D(p,q)$, $v \in J(p,q)$, or $v \in D(q,p)$
- Given a point v in common to three bisecting curves, determine the clockwise order of the curves around v
- Given points $u \in J(p,q)$ and $w \in J(p,r)$ and orientation of these curves , determine the first point of $J(p,r) \mid_{(w,\infty]}$ crossed by $J(p,q)|_{(v,\infty]}$
- Given $J(p,q)$ with an orientation and points v, w, x on $J(p,q)$, determine if v come before w on $J(p,q) \mid_{(x,\infty]}$

Notation: Give a connected subset A of R^2, int$A$, bd$A$, and cl$A$ mean the interior, the boundary, and the closure of A, respectively.

Conflict Graph $G(R)$, where R is R_i for $2 \leq i \leq n$

- bipartitle graph (U, V, E)
- U: Voronoi edges of $V(R)$
- V: Sites in $S \setminus R$
- $E : \{(e, s) \mid e \in V(R), s \in S \setminus R, e \cap \mathrm{VR}(s, R \cup \{s\}) \neq \emptyset\}$
  - a conflict relation beteween e and s.

Remark:
a Voronoi edge is defined by 4 sites under the general position assumption



Lemma 1

Let $R \subseteq S$ and $t \in S \setminus R$. Let e be the Voronoi edge between $\mathrm{VR}(p, R)$ and $\mathrm{VR}(q, R)$. $e \cap \mathrm{VR}(t, R \cup \{t\}) = e \cap \mathrm{R}(t, \{p, q, r\})$. (Local Test is enough)

Proof:

$\subseteq$: Immediately from $\mathrm{VR}(t, R \cup \{t\}) \subseteq \mathrm{VR}(t, \{p, q, t\})$

$\supseteq$: Let $x \in e \cap \mathrm{VR}(t, \{p, q, t\})$

- Since $x \in e$, $x \in \mathrm{VR}(p, R) \cup \mathrm{VR}(q, R)$ and $x \notin \mathrm{VR}(r, R) \supseteq \mathrm{VR}(r, R \cup \{t\})$ for any $r \in R \setminus \{p, q\}$.
- Since $x \in \mathrm{VR}(t, \{p, q, t\})$, $x \notin \mathrm{VR}(p, \{p, q, t\}) \cup \mathrm{VR}(q, \{p, q, t\}) \supseteq \mathrm{VR}(p, R \cup \{t\}) \cup \mathrm{VR}(q, R \cup \{t\})$
- $x \notin \mathrm{VR}(r, R \cup \{t\})$ for any site $r \in R \to x \in \mathrm{VR}(t, R \cup \{t\})$

Insertiong $s \in S \setminus R$ to compute $V(R \cup \{s\})$ and $G(R \cup \{s\})$ from $V(R)$ and $G(R)$. Handle a conflict between s and a Voronoi edge e of $\mathrm{VR}(R)$

Lemma 2

$\mathrm{cl}\ e \cap \mathrm{cl}\ \mathrm{VR}(s, R \cup \{s\}) \neq \emptyset$ implies $e \cap \mathrm{VR}(s, R \cup \{s\}) = \emptyset$

proof

- Let x belong to $\mathrm{cl}\ e \cap \mathrm{cl}\ \mathrm{VR}(s, R \cup \{s\})$
- x is an endpoint of e:
  - x is the intersection among three curves in R
  - For any $r \in R$, $J(s, r)$ cannot pass through x due to the general position assumption
  - $x \in D(s, r) \to$ the neighborhood of $x \in D(s, r)$
  - $\exists y \in e$ belongs to $\mathrm{VR}(s, R \cup \{s\})$
- $x \in e \cap \mathrm{bd}\ \mathrm{VR}(s, R \cup \{s\})$
  - $x \in J(p, q) \cap J(s, r)$
  - a point $y \in e$ in the neighborhood of x such that $y \in \mathrm{VR}(s, R \cup \{s\})$

Theorem 2

$V(S)$ can be computed in $O(nlogn)$ expected time

- $\Sigma_{3\leq i\leq n} O(\Sigma_{(e,s_i)\in G(R_{i-1})} \deg_{G(R_{i-1})}(e))$
- Let e be a Voronoi edge of $V(R_i)$ and let s be a site in $S \setminus R_i$ which conflicts e.
- The conflict relation (e, s) will be counted only once since the counting only occured when e is removed
  - Let s_j be the earliest site in the sequence which conflicts with e. Then (e, s) will be counted in $\deg_{G(R_{j-1})}(e)$
- Time proportional to the number of conflict relations between Voronoi edges in $\cup_{2\leq i\leq n} V(R_i)$ and sites in S
- The expected size of conflict history is $-C_n + \Sigma_{2\leq i\leq n}(n - j + 1)p_j$
  - C_n is the expected size of $\cup_{2\leq i\leq n} V(R_i)$
  - p_j is the expected number of Voronoi edges defined by the same two sites in $V(R_j)$
- Since $C_n = O(n)$ and $p_j = O(1/j)$, the expected run time is $O(n \log n)$