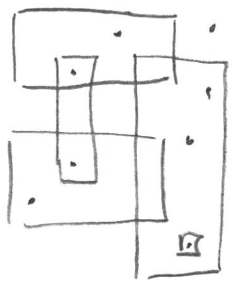
Computation of smallest transversal can be difficult:

Example 1: R = finite set of (axis-parallel) rectangles in $\mathbb{R}^2$

P = finite set of points



Problem: Find minimal subset $P_{min} \subseteq P$, so that any $r \in R$ contains a point in P_{min}.

NP-complete (also for axis-parallel lines !)

Transversal: $X := P$ $\mathcal{F} := \{ P \cap r \mid r \in R \}$

Example 2: R = finite sets of intervals along X-axis

P = finite set of points

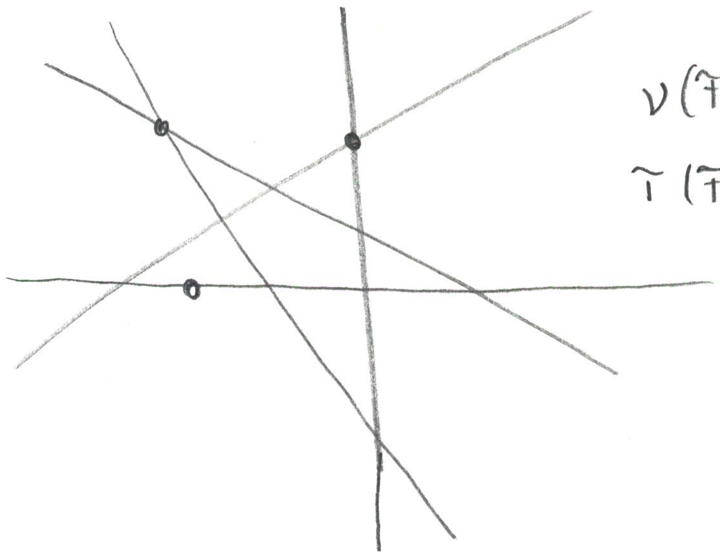
Problem: Find minimal subset $P_{min} \subseteq P$ so that any $I \in P$ contains at least one point.

packing / transversal

Does always $\nu(\mathcal{F}) = \tau(\mathcal{F})$ hold? (101)

No, example $\nu < \tau$

$X = \mathbb{R}^2$, $\mathcal{F} = \{n \text{ lines in general position}\}$



$\nu(\mathcal{F}) = 1$

$\tau(\mathcal{F}) = \lceil \frac{n}{2} \rceil$

($\nu \geq \tau$ not in general)

Proposition: $\nu(\mathcal{F}) \leq \tau(\mathcal{F})$

Proof: For m disjoint sets of $\mathcal{F}$ at least m stabbing points are necessary.

Relaxation of the transversal $\rightarrow$ ε-net

Large sets should be easier hit by a transversal than small ones!

Simple, special case X is finite

Size of a set by cardinality.

<u>Definition 48</u> (ε-net, special case)

Let $(X, \mathcal{F})$ be a set system an $\varepsilon \in [0,1]$ a real number. A set $N \subseteq X$ (not necessarily $N \in \mathcal{F}$!) is called an ε-net for $(X, \mathcal{F})$ if $N \cap S \neq \emptyset$ for all $S \in \mathcal{F}$ with $|S| \geq \varepsilon |X|$.

- ε-net is transversal for all sets larger than $\varepsilon \cdot |X|$
- Convenient to write $\varepsilon = \frac{1}{r}$ for $r \geq 1$ $r \in \mathbb{R}$.
- Epsilon net Theorem $\rightsquigarrow$ Condition on $\mathcal{F} \Rightarrow$ existence of $\frac{1}{r}$ nets of size $O(r \log r)$
- Art gallery: all vispolygons have size larger than $\frac{1}{r}$ vol(P) ✓
  \# guards in $O(r \log r)$ ✓

$X = P$ not finite

general definition, μ a probability measure

in most cases μ is volume

or μ gives cardinality ✓ (finite sets)

P polygon:

$$\mu(P) = 1 \qquad \mu(\text{vis}_P(p)) = \frac{\text{vol}(\text{vis}_P(p))}{\text{vol}(P)} > \varepsilon = \frac{1}{r}$$

Definition 43 (ε-net)

$N \subseteq X$ is called ε-net for $(X, \mathcal{F})$

$:\Leftrightarrow \forall F \in \mathcal{F} \ (\mu(F) \geq \varepsilon \Rightarrow F \cap N \neq \emptyset)$.

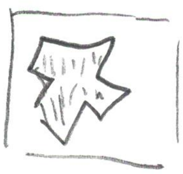
In other words: N is a transversal of a subsystem of $\mathcal{F}$ with sufficient large sets.

Fits to our art gallery question: $\mu(\text{vis}_P(p)) \geq \frac{1}{r} \mu(P)$
$\forall p \in P$

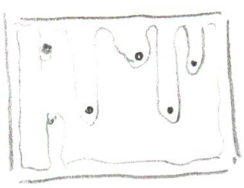
Searching for sufficient conditions for the existence of finite (small) ε-nets. VC-dim

Examples 1 $\quad X$ = unit square $Q = [0,1]^2$

$\mathcal{F} = \{ F \subset Q \text{ area of simple polygon} \}$



No finite ε-net exists:

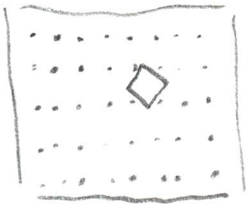


Assume finite net!
There are always polygons with volume arbitrarily close to 1 that do not contain the net points.

Example 2 $X = Q$ unit square

$$\mathcal{F} = \{ F \subset Q \text{ square} \}$$

For all $\varepsilon > 0$ there is a finite ε-net (depending on ε)

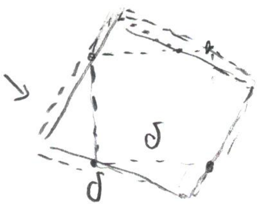


Grid with distance δ

ε-net for $(X, \mathcal{F})$ with $\varepsilon = 2\delta^2$

unit square that is hidden between grid points

maximal size



rectangle $\sqrt{2} \cdot d$ side length.

Theorem 44 (ε-net Theorem): Let $(X, \mathcal{F})$ be a set system with measure μ let $\dim_{VC}(\mathcal{F}) = d < \infty$.

For $r > 2$ there is an ε-net for $\mathcal{F}$ with size at most $C \cdot d\, r \ln r$, where C is an independent constant.

Bound on C: Fundamental lemma bounding the number of distinct sets in a system of given VC dimension.

Definition 45

Let $(X, \mathcal{F})$ be a set system $\quad \mathcal{F} \subseteq P(X)$

$$\pi_{\mathcal{F}}(m) := \max_{\substack{Y \subseteq X \\ |Y| = m}} |\mathcal{F}|_Y|$$

is denoted as the shatter function of $(X, \mathcal{F})$.

[$\pi_{\mathcal{F}}(m)$ maximum possible number of distinct intersection of the sets $\mathcal{F}$ with an m-point subset. $Y \in \mathcal{F}$ is not required, and also that Y is fully shattered]

$$\mathcal{F}|_Y = \{F \cap Y \mid F \in \mathcal{F}\}$$

Relationship between $\pi_{\mathcal{F}}(m)$ and $\dim_{VC}(\mathcal{F})$

Clear: $\quad \dim_{VC}(\mathcal{F}) = \infty \quad \Leftrightarrow \quad \forall m: \ \pi_{\mathcal{F}}(m) = 2^m$

(any subset intersected with different sets)

"$\Leftarrow$" any m set is fully shattered, Def VC-dim

"$\Rightarrow$" Let $m \leq n$ $\quad \overset{|B|=n}{|\mathcal{F}|_B| = 2^n}$ exists $\dim_{VC}(\mathcal{F}) = \infty$

Let $A \subseteq B \quad |A| = m$

$\Rightarrow |\mathcal{F}|_A| = 2^m \Rightarrow \pi_{\mathcal{F}}(m) = 2^m$ for all m.

Lemm 46: (Shatter function lemma) X finite

(i) Let $|X| = m$ and $\dim_{VC}(\mathcal{F}) \leq d$

Then $\mathcal{F}$ consists of at most

$\binom{m}{0} + \binom{m}{1} + \dots + \binom{m}{d}$ subsets of X

(ii) For $\dim_{VC}(\mathcal{F}) \leq d$

$$\Pi_{\mathcal{F}}(m) \leq \binom{m}{0} + \binom{m}{1} + \dots + \binom{m}{d}$$

(iii) $\binom{m}{0} + \binom{m}{1} + \dots + \binom{m}{d} \leq \left(\frac{em}{d}\right)^d \in O(m^d)$

Proof:

First $(X, \mathcal{F})$ and $X' \subseteq X$

$\Rightarrow \dim_{VC}(\mathcal{F}|X') \leq \dim_{VC}(\mathcal{F})$

Let $C \subseteq X'$ assume C is shattered by $\mathcal{F}|_{X'}$

$\Rightarrow \forall B \subseteq C \; \exists \; F' \in \mathcal{F}|X' : B = C \cap F'$

$\Rightarrow \forall B \subseteq C \; \exists \; F \in \mathcal{F} \quad B = C \cap (F \cap X') \underset{C \subseteq X'}{=} C \cap F$

$\Rightarrow C$ is shattered by $\mathcal{F}$

Part (i). Induction over d, nested induction on m

$d \leq m$ clear no more than $|X|$ elements.

Ind. base $d = 0$ arbitrary m

$\dim_{VC}(\mathcal{F}) = 0 \Rightarrow$ no set of one element is shattered

$\Rightarrow a \in X \quad \{\}, \{a\} \subseteq \{a\}$

either no $F \in \mathcal{F}$ with $F \cap \{a\} = \{a\}$ or $F \cap \{\} = \{\}$ ✓

$\Rightarrow F \cap \{a\} = \phi \;\; \forall F \; \forall a \;\Rightarrow\; F = \{\}$

$\Rightarrow |\mathcal{F}| = 1 = \binom{m}{0} = \binom{m}{d}$

Ind. step $d \geq 1$ (also $m \geq 1$) let $a \in X$.

$X_1 := X \setminus \{a\} \qquad \mathcal{F}_1 := \mathcal{F}|X_1 = \{F \cap X_1 \mid F \in \mathcal{F}\}$

$\dim_{VC}(\mathcal{F}_1) \leq \dim_{VC}(\mathcal{F}) \leq d$

Induction on m $\quad |\mathcal{F}_1| \leq \binom{m-1}{0} + \binom{m-1}{1} + \dots + \binom{m-1}{d}$

How many sets can $\mathcal{F}$ have more than $\mathcal{F}_1$

Consider the mapping

$f : \mathcal{F} \to \mathcal{F}_1$

$A \mapsto A \setminus \{a\}$

We have

$$f(A_1) = f(A_2) \iff A_2 = A_1 \dot{\cup} \{a\}$$

$A_1 \neq A_2$ or $A_1 = A_2 \dot{\cup} \{a\}$ — one set has one element more

Let $\tilde{F}_2 := \{A \in \tilde{F} \mid a \notin A \text{ and } A \dot{\cup} \{a\} \in \tilde{F}\}$ setsystem over X_1

an element of A_1, A_2 with the smaller number of elements

$$\Rightarrow |\tilde{F}| = |\tilde{F}_1| + |\tilde{F}_2|$$

because

$|\tilde{F}_2|$ number of sets with "a" or without "a" (has counterpart)

plus $|\tilde{F}_1|$ counterparts to $\tilde{F}_2$ with $a \in A'$

and sets in $\tilde{F}$ with $F \cap X_1 = F$

Claim: $\dim_{VC}(\tilde{F}_2) \leq d-1$, setsystem over X_1

Proof. Show: $\forall A \subseteq X_1$ A shattered by $\tilde{F}_2 \overset{!}{\Rightarrow} A \dot{\cup} \{a\}$ shattered by $\tilde{F}$

If this is true we have

$$|A| < |A|+1 = |A \dot{\cup} \{a\}| \leq \dim_{VC}(\tilde{F}) \leq d$$

$$\Rightarrow |A| \leq d-1$$

Now let $A \subseteq X_1$, $B \subseteq A \dot{\cup} \{a\}$.

Case I. $a \notin B$: $B \subseteq A \underset{(A \text{ sh. by } \tilde{F}_2)}{\Rightarrow}$ ex $F \in \tilde{F}_2$ with $F \cap A = B$

$\tilde{F}_2 \subseteq \tilde{F}$ $\Rightarrow F \cap A \dot{\cup} \{a\} = B$

$\mathcal{F}_1 := \mathcal{F} | X_1$

$X_1 := X \setminus \{a\}$ $\quad |X_1| < m$ ind hyp.

$\mathcal{F}_2 := \{ A \in \mathcal{F} \mid a \notin A \text{ and } A \,\dot{\cup}\, \{a\} \in \mathcal{F} \}$

$\mathcal{F} = |\mathcal{F}_1| + |\mathcal{F}_2|$ $\quad \dim_{VC}(\mathcal{F}_2) \leq d-1$ ind. hyp