Probability of all three events!

Formally Bernoulli-Experiment (Stochastik)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Set of events

Events A, B on Ω with $A \cup B \subseteq \Omega$

(Size, height)

$$P(\Omega) = 1$$

$$1 \geq P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) \geq P(A) + P(B) - 1$$

Events (Sets) A, B, C

A height	$\leq C \log n$	$\geq \frac{3}{4}$
B size	$\leq C\, n$	$\geq \frac{3}{4}$
C constr. time	$\leq C\, n \log n$	$\geq \frac{3}{4}$

Apply twice!

$$P[A \cap B \cap C] \geq P[A] + P[B \cap C] - 1$$

$$\geq P[A] + P[B] + P[C] - 2$$

$$\Rightarrow \quad 3 \cdot \frac{3}{4} - 2 = \frac{1}{4}$$

The number of expected repetitions for the "algorithm":

$$\binom{i}{1} p^1 \cdot (1-p)^{i-1}$$

Probability of 1st success for i tries!

$$= i \cdot \frac{1}{4} \left(\frac{3}{4}\right)^{i-1}$$

$\left[\binom{n}{k} \left(p^k \, q^{n-k} \right) \right.$

dices 10 throws
3 sixes $p = \frac{1}{6}$

$$p^3 \cdot (1-p)^{(10-3)}$$

first the sixes than no sixes!
Indep. – Order $\Rightarrow$ times $\binom{n}{k}$ $\left. \right]$

Expected value

$$\leq \sum_{i=1}^{\infty} i \cdot \frac{1}{4} \cdot \left(\frac{3}{4}\right)^{i-1}$$

$$= \frac{1}{4} \underbrace{\left(\sum_{i=1}^{\infty} i \cdot x^{i-1} \right)_{x=\frac{3}{4}}}$$

Series within its convergence radius

$$= \frac{1}{4} \left(\frac{1}{(x-1)^2} \right)_{x=\frac{3}{4}} = 4$$

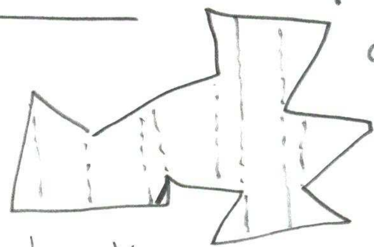
$\square$

Now back to expected size/height/running time!

Line segments that build a simple polygon!

(R Seidel CGTA 1991)

Trapezoidal decomp.



Localization of new segments costly!

Idea: Chain ⇝ use the next end-point (easy to localize)
trace!

Problem: Random choice is destroyed!

Solution: - Random insertion as before but
- Trace the chain from time to time
in order to localize end-points

This helps as follows.

Lemma 53: For $1 \leq j \leq k \leq n$.

If q is a query point for which the trapezoid of $T(S_j)$ is known we can localize q in $T(S_k)$

in average (expected time)

$$O\left(\frac{1}{j+1} + \frac{1}{j+2} + \cdots + \frac{1}{k}\right) \in O\left(\log \frac{k}{j}\right)$$

Proof: As in the proof of Theorem 49

backward analysis starts at $T(S_j)$

probab. that s_{j+1} changes trapezoid of $T(S_{j+1})$ is $\frac{4}{j+1}$!

and so on !

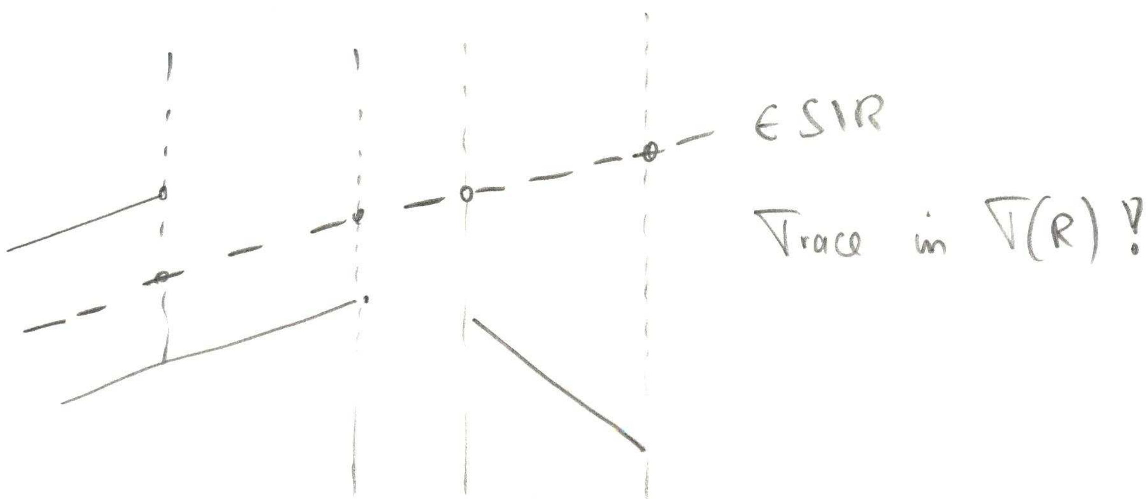
$$\sum_{i=j+1}^{k} 3 \cdot \frac{4}{i} \in O\left(\log \frac{k}{j}\right)$$

$\square$

Estimation of the cost of a trace ! Trace $S \setminus R$ through $T(R)$

Lemma 54:

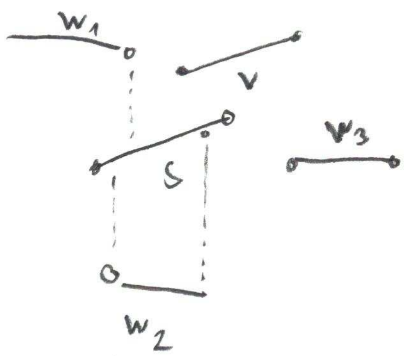
Let $R \subset S$ be a subset of non intersecting segments (P is a single polygon). Then the segments of $S \setminus R$ have (in the average) at most $O(|S|-|R|)$ intersections with the vertical lines of $T(R)$.



Proof:

For $s \in W \subset S$ let

$\deg(s,W) :=$ # vertical prolongations of segments in W that hit s



Clear: $\sum_{s \in W} \deg(s,W) \le 4|W|$

(any segments gets at most 4 hits v)

$$\frac{1}{\binom{n}{r}} \sum_{\substack{R \subset S \\ |R| = r}} \underbrace{\sum_{s \in S \setminus R} \deg(s, R \cup \{s\})}_{\le 4(r+1)}$$

(average)

$$= \frac{1}{\binom{n}{r}} \underbrace{\sum_{\substack{R' \subset S \\ |R'| = r+1}}}_{\binom{n}{r+1} \text{ summands}} \underbrace{\sum_{s \in R'} \deg(s, R')}_{\le 4(r+1)}$$

$$\le \frac{1}{\binom{n}{r}} \binom{n}{r+1} \cdot 4(r+1) = \frac{4\cancel{(r+1)}\,\cancel{n!}\,\cancel{r!}\,(n-r)!}{\cancel{(r+1)}!\,(n-r-1)!\,\cancel{n!}}$$

$= 4(n-r)$ $\square$

Definition 55 ($\log^{(i)} n$, $\log^* n$)

$$\log^{(i)} n := \underbrace{\log(\log(\ldots(\log n)\ldots))}_{i\text{-times}}$$

$\log^* n :=$ smallest h so that $\log^{(h)} n \leq 1$.

(Remark grows very, very slowly)

Algorithm: Segments $s_1, \ldots, s_n$

Phases: $1, \ldots, \frac{n}{\log n}, \ldots, \frac{n}{\log\log n}, \ldots, \frac{n}{\log^{(3)} n}, \ldots, \frac{n}{\log^{(\log^* n)} n}, \ldots, n$

Choose insertion order permutation randomly $s_1, \ldots, s_n$

For any phase: $\frac{n}{\log^{(s-1)} n}, \ldots, \frac{n}{\log^{(s)} n}$

- insert segments with corresp. indices
- trace at the end of the phase ∂P through the last structure. Localize all end-points of non-inserted segments!

Theorem 56

A trapezoidal decomposition and a query structure of a simple polygon can be computed in expected $O(n \log^* n)$ time. (instead of $O(n \log n)$ for arbitrary segments)

Proof:

$\log^* n$ - many parts

$\Rightarrow O(n \log^* n)$ for all traces!

(Lemma 54)

Insertion of a segment

Searching for the first endpoint

(Lemma 53)

$$O\left(\log \frac{\frac{n}{\log^{(h)} n}}{\frac{n}{\log^{(h-1)} n}}\right) = O\left(\log \frac{\log^{(h-1)} n}{\log^{(h)} n}\right)$$

$$= O\left(\log^{(h)} n + \log \frac{1}{\log^{(h)} n}\right)$$

$$\in O\left(\log^{(h)} n\right)$$

Follow the segments

As before (Theorem 49 $E(X_i)$)

$$\left[\begin{array}{l}\# \text{ of new trapezoids in} \\ T(S_i)\end{array}\right]$$

$E(X_i) \in O(1)$

$$\underset{\text{one part}}{\sum \text{costs}} \leq \frac{n}{\log^{(h)}} \cdot O(\log^{(h)} n) \in O(n)$$

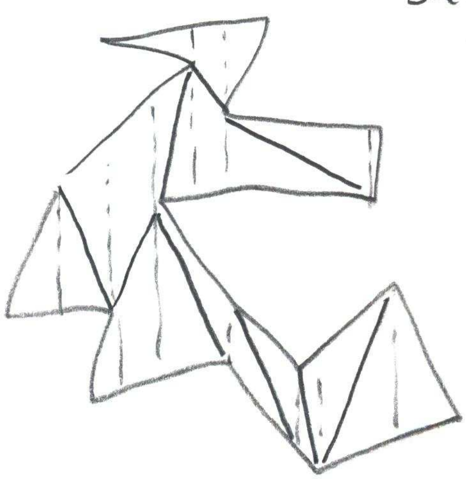
$\log^* n$ many parts

$\Rightarrow$ $O(n \log^* n)$ construction cost (expected) ☐

$\left\{\begin{array}{ll}\text{Expected Query time} & O(\log n) \\ \text{Expected Size} & O(n)\end{array}\right.$ Th. 49 $\}$

Theorem 57. A simple polygon can be triangulated in expected $O(n \log^* n)$ time.

Sketch.



Trapezoidal map:

+ Insert edge, vertices on different sides
+ Remaining polygons two x-monotone chains. (one is a single segment)
+ Easy to triangulate Successively delete convex vertices

☐