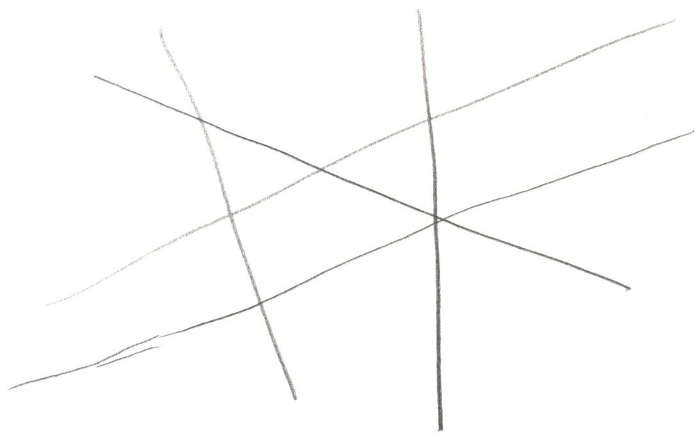
Introduction

Discrete and Computational Geometry

↳ Discrete sets: points, lines, circles, ... $\in \mathbb{R}^d$

Examples: Structural Properties

I. n lines in the plane



Q: How many regions?

II. n points in the plane

Q: How many of them can have the same distance?

Main reference: J. Matoušek Lectures on Discrete Geometry

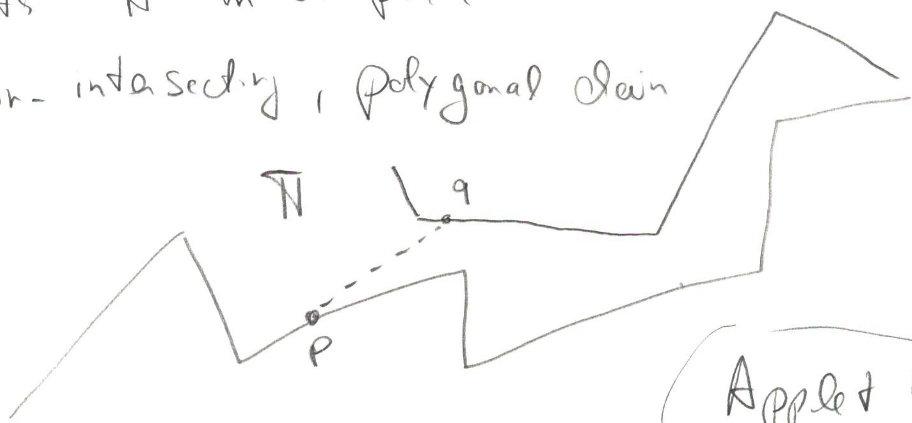
Computational Geometry

Algorithms for solving geometric problems

Example:

Path π in the plane

non-intersecting, polygonal chain



Applet!

Find points p, q with

$$\frac{|\pi_p^q|}{|pq|} = \max_{r,s \in \pi} \frac{|\pi_r^s|}{|rs|}$$

Maximal detour pair of points

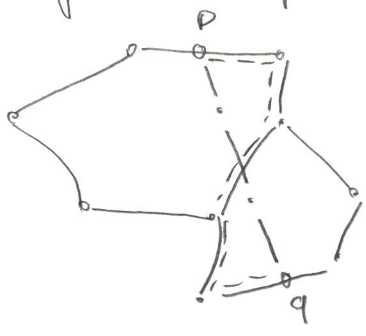
Computational time: $O(n \log n)$

Starting point! Scientific research articles!

[Black Boxes Sometimes]

Dilation of a chain in $\mathbb{R}^2$

More general: planar Graph $G = (V, E)$



Definition 0

Geometric dilation of G

$$\delta_{geom}(G) := \sup_{p \neq q} \frac{|\pi_p^q|}{|pq|}$$

worst-case detour of the network

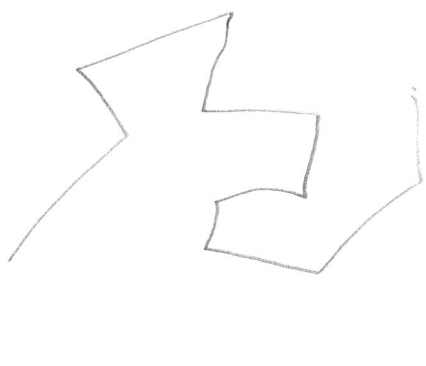
Graph-theoretic dilation

$$\delta_{graph}(G) := \sup_{\substack{p \neq q \\ p, q \in V}} \frac{|\pi_p^q|}{|pq|}$$

detour between major points

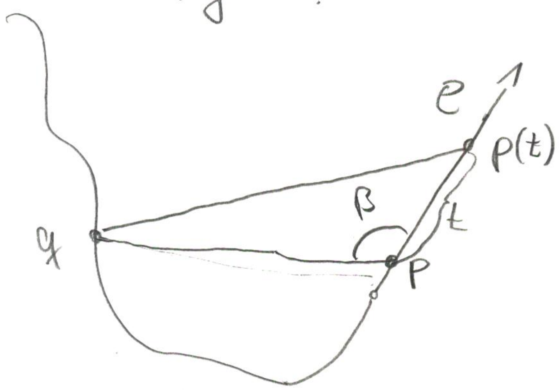
Natural questions: Compute $\delta(G)$!

Now: G is a polygonal chain: C



Try to find structural properties for the worst-case pair (p, q)

Figure 1.



Fix

$q \in C$

$p \in e$, $p(t) \in e$

Where is the maximum? Derivative! Local maxima!
(along e)

$\delta(p(t), q)$ versus $\delta(p, q)$

$$\delta(p(t), q) = \frac{t + |C_p^q|}{\sqrt{t^2 - 2t\,|pq| \cos\beta + |pq|^2}}$$

Law of cosine

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{f^2}$$

$$\delta'(p(t), q) = \frac{\sqrt{\ldots} - (t + |C_p^q|)\,\frac{1}{2}\,\frac{1}{\sqrt{\ldots}}\,(2t - 2|pq| \cos\beta)}{\sqrt{\ldots}^{\,2}}$$

Derivation $t = 0$ $\quad$ $\delta'(p(t), q) > 0$ increase

Tangent $\quad$ $\delta'(p(t), q) < 0$ decrease

$\delta'(p(t), q) = 0$ local Extremum

$$t=0 \implies \frac{|pq|^2 - |C_p^q|\left(-|pq|\cos\beta\right)}{|pq|^2}$$

$$= 1 + \frac{|C_p^q|}{|pq|}\cos\beta \quad \left\{ \begin{array}{l} >0 \\ <0 \\ =0 \end{array} \right.$$

Lemma 1: The local dilation in Figure 1

decreases if we move p toward $p(t)$ $\iff$ $\cos\beta < -\frac{|pq|}{C_p^q}$

attains a local maximum at p $\iff$ $\cos\beta =$ ⋮

increases if we move p toward $p(t)$ one $\iff$ $\cos\beta >$ ⋮

Proof: Just shown. □

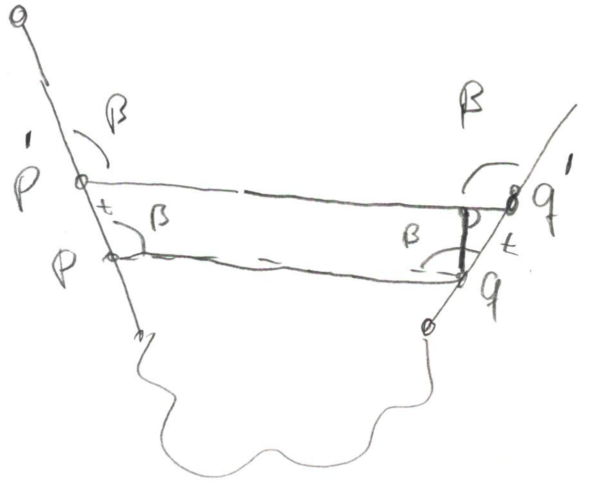
A direct consequence:

Lemma 2: The maximal dilation of a polygonal chain is attained for a pair (p,q) where at least one point is a vertex.

Proof: Let (p,q) be a pair that maximizes the dilation.

Assume that neither p nor q is a vertex.

Due to Lemma 1 (Optimality on both sides) we have the following situation:

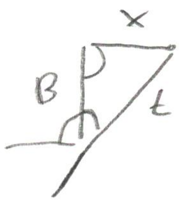


with $\cos\beta = -\frac{|pq|}{|C_p^q|}$

(Covers also the V situation)

Construction: Shift pq "up" until a vertex is attained. Does not change the dilation:

$$\delta(p',q') = \frac{2t + |C_p^q|}{|pq| - 2t\cos\beta} = \frac{2t + |C_p^q|}{|pq| + 2t\frac{pq}{|C_p^q|}} = \frac{|C_p^q|}{|pq|} = \delta(p,q)$$



$$\frac{x}{\sin\beta - \frac{\pi}{2}} = \frac{t}{\sin\frac{\pi}{2}} \iff x = t\sin\left(\beta - \frac{\pi}{2}\right) = t(-\cos(\beta))$$

□

Direct consequence:

$\delta(C)$ can be computed in $O(n^2)$ time

Compute $|C_{p_0}^{p_i}|$ in linear time

Compute $|C_{p_i}^{p_8}|$ in constant time

Compute for any edge-vertex pair $\delta(p,e)$ in $O(1)$

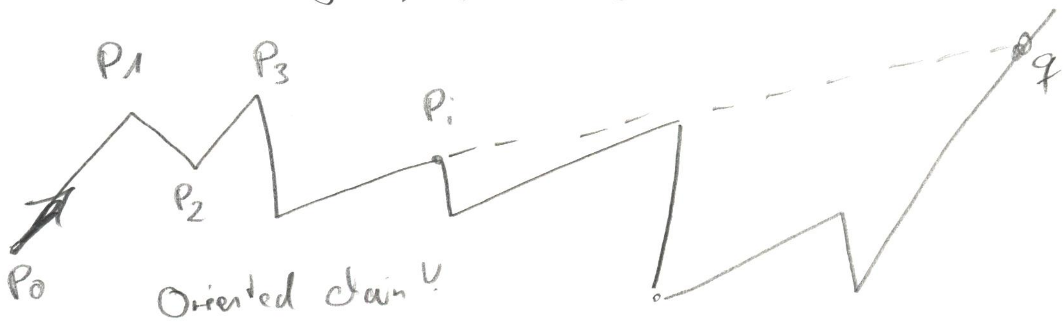
$\Rightarrow O(n^2)$!

Improvements !

Decision problem (easier to solve?)

C and number $K \geq 1$ given:

$$\delta(C) \leq K \ ?$$



Oriented chain!

Check the following in both directions:

Does $\delta(p_i, q) \leq K$ hold for all q "after" p_i ?

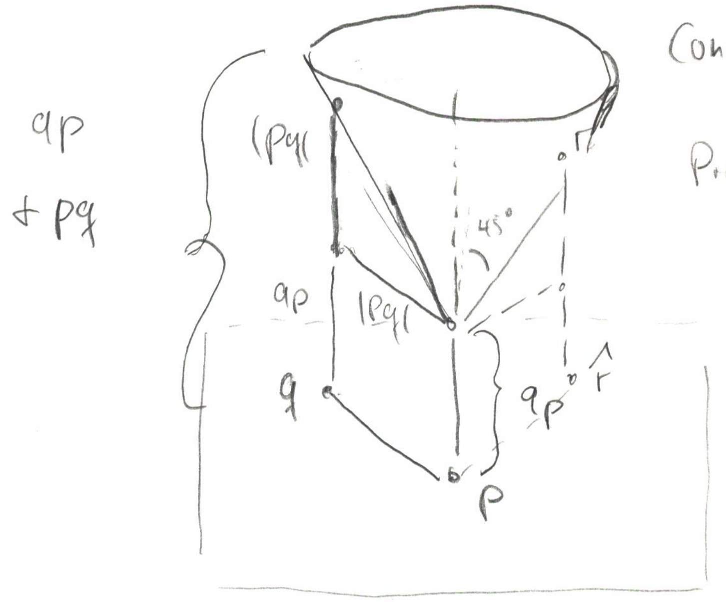
<u>Trick:</u> $\delta(p,q) \leq K$

$$\Longleftrightarrow \frac{|C_p^q|}{|pq|} = \frac{|C_{p_0}^q| - |C_{p_0}^p|}{|pq|} \leq K$$

$$\Longleftrightarrow \underbrace{\frac{|C_{p_0}^q|}{K}}_{=:\, a_q} \leq |pq| + \underbrace{\frac{|C_{p_0}^p|}{K}}_{=:\, a_p}$$

This has a nice geometric interpretation!

$$a_p := \frac{|C^{p}_{p_0}|}{\mathcal{K}}$$



Cone K_p at height a_p over p ✓

Projections of points on K_p to the plane!

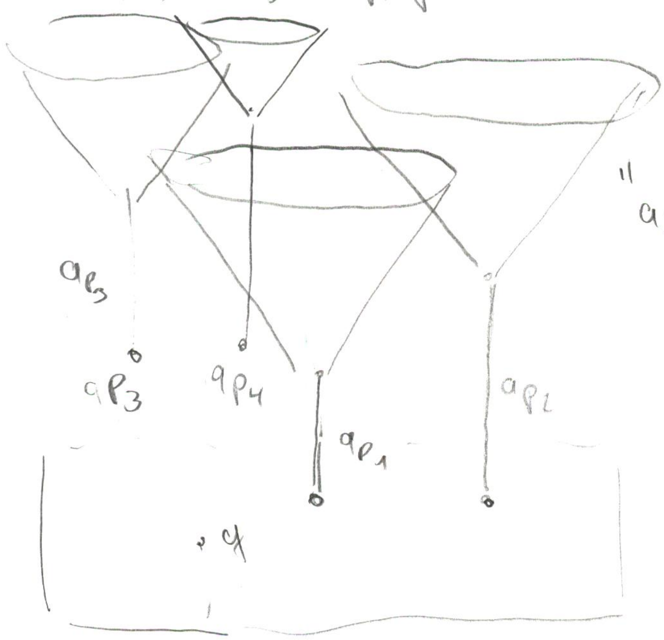
$$|r\hat{r}| = a_p + |p\hat{r}|$$

We have

(*) $\delta(p,q) \leq \mathcal{K} \iff a_q \leq |pq| + a_p$

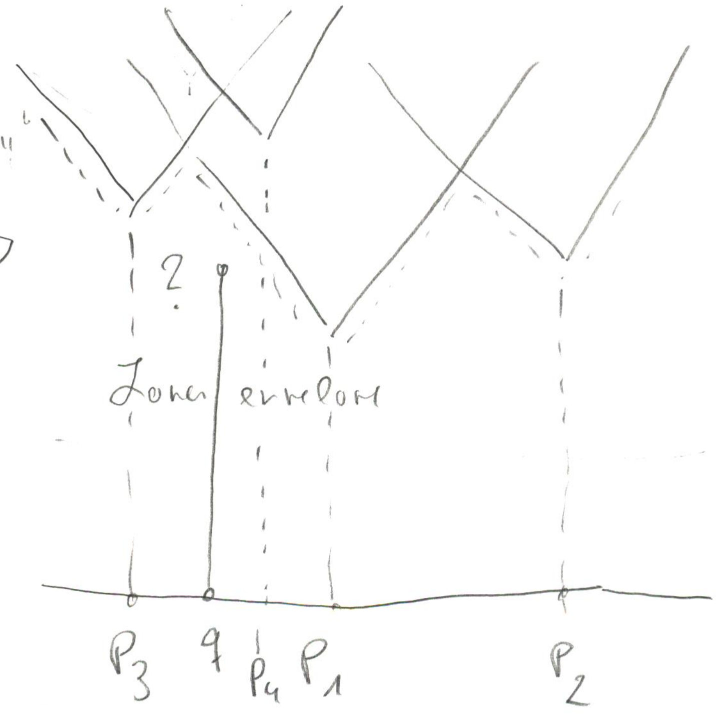
$\iff$ point (q_x, q_y, a_q) is below K_p

This must hold for all vertices p before q!



More cones!

"a slice" $\Longrightarrow$



(Simplification)