Recap 30.4.2012

Szemerédi Trotter:

Incidences of points and lines $\mathbb{R}^2$

$T(n,n) \in \Omega(n^{\frac{4}{3}})$

$T(n,m) \in O(n^{\frac{2}{3}} m^{\frac{2}{3}} + m + n)$

via crossing numbers of Graphs

Crossing number Theorem: $G=(V,E)$ simple

$$cr(G) \geq \frac{1}{64} \frac{|E|^3}{|V|^2} - |V|$$

maximal planar $3|V|-6 \geq |E|$

at least $|E|-3|V|$ crossings

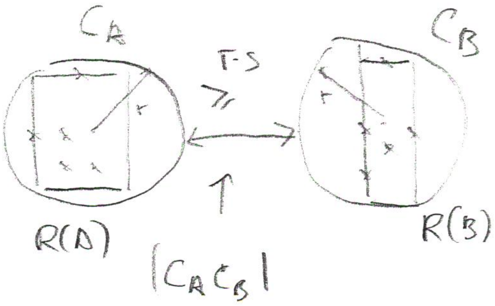
Probabilistic argument x' crossing number of drawing

$$x' \geq m' - 3n'$$

$T(n,m) \in O(\dots)$ Graph construction!

WSPD: Applications Distance problems
Spanner Construction

well separated w.r.t s



1. $C_A \cap C_B = \phi$
2. $R(A) \subseteq C_A$, $R(B) \subseteq C_B$
3. $|C_A C_B| \geq s \cdot r$

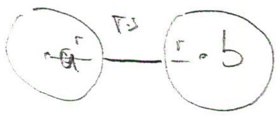
WSPD: $S \subseteq \mathbb{R}^d$

$(A_1,B_1),(A_2,B_2),\dots,(A_m,B_m)$ $\quad A_i,B_i \subseteq S$

- A_i, B_i well-sep. w.r.t. s $\quad 1\le i\le m$
- $p\neq q \in S$ unique i so that $p\in A_i$ and $q\in B_i$ or $q\in A_i$ and $p \in B_i$

Example: $S\subseteq\mathbb{R}^d$

$(\{a\},\{b\})$



$\binom{n}{2}$ pairs

$|ab| = r + r\cdot s + r = (2+s)r$

$r = \frac{|ab|}{2+s}$

all well-sep. $\rightsquigarrow$ Appr. closest pair

Lemma $a,a'\in A$ $\quad b,b'\in B$

$|a a'| \le \frac{2}{s}|ab|$

$|a'b'| \le \left(1+\frac{4}{s}\right)|ab|$

<u>Theorem</u> 19 For a set S of n points in $\mathbb{R}^d$ and a parameter s, a WSPD of S with $m \in O(s^d d^{\frac{d}{2}} n)$ many pairs can be computed in time $O(dn \log n + s^d \cdot d^{\frac{d}{2}} \cdot n)$.

$\left(s,d \text{ fixed: size } O(n) \text{ time } O(n \log n) \right)$

Does not necessarily mean $\sum_{i=1}^{m} |A_i| + |B_i| \in O(n)$!

Larger than $\binom{n}{2}$!

Application: Spanner construction!

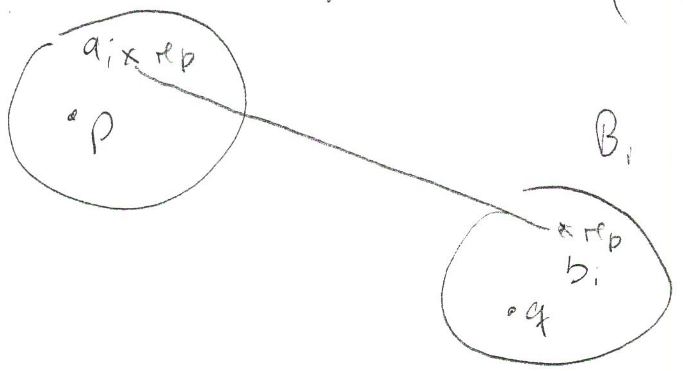
<u>Definition 20</u> Let S be a set of n points in $\mathbb{R}^d$ and $t \geq 1$, $t \in \mathbb{R}$.

A t-spanner for S is a graph G with vertex set S, so that for any two points p and q of S there is a path $\Pi_{G_p}^q$ in G with $|\Pi_{G_p}^q| \leq t\,|pq|$.

<u>t-spanner path</u>

Lemma 21 Let a WSPD be given for $s > 4$.

Take one (arbitrary) edge for each pair of the decomposition. This gives a t-spanner with $t = \frac{s+4}{s-4}$ with m edges.



Proof: Induction on the rank of the distances $|pq|$

Ind. base $|pq| = 0$ ✓

Ind. step $|pq| > 0$ w.l.o.g.

$$p \in A_i \quad q \in B_i \quad \text{(def. WSPD)}$$

Ind. hypothesis: For any two points x and y in S with $|xy| < |pq|$ the graph (S, E) has a t-spanner path between x and y

Lemma 14 :) $\quad |pa_i| \leq \left(\frac{2}{s}\right) |pq| < |pq|$

$\Uparrow$ rep. $\qquad$ $s > 4$

$\Rightarrow$ t-spanner path P_1 in G between p and a_i
Ind.-hyp.

Similarly, $|b_i q| < |pq|$ and path P_2 in G

Path P: P_1 conc. $\{a_i, b_i\}$ conc. P_2

$$|P| \leq t\,|pa_i| + |a_i b_i| + t\,|b_i q|$$

Lemma 14 ii) $|a_i b_i| \leq \left(1 + \frac{4}{s}\right)|pq|$

$$|P| \leq \left(\left(\frac{2}{s} + \frac{2}{s}\right)t + 1 + \frac{4}{s}\right)|pq|$$

$$= \left(\frac{4(t+1)}{s} + 1\right)|pq| = t\,|pq|$$

$$\left(t = \frac{s+4}{s-4} \iff (s-4)t = (s+4) \iff s(t-1) = 4(t+1) \iff s = \frac{4(t+1)}{(t-1)} \right)$$

$\square$

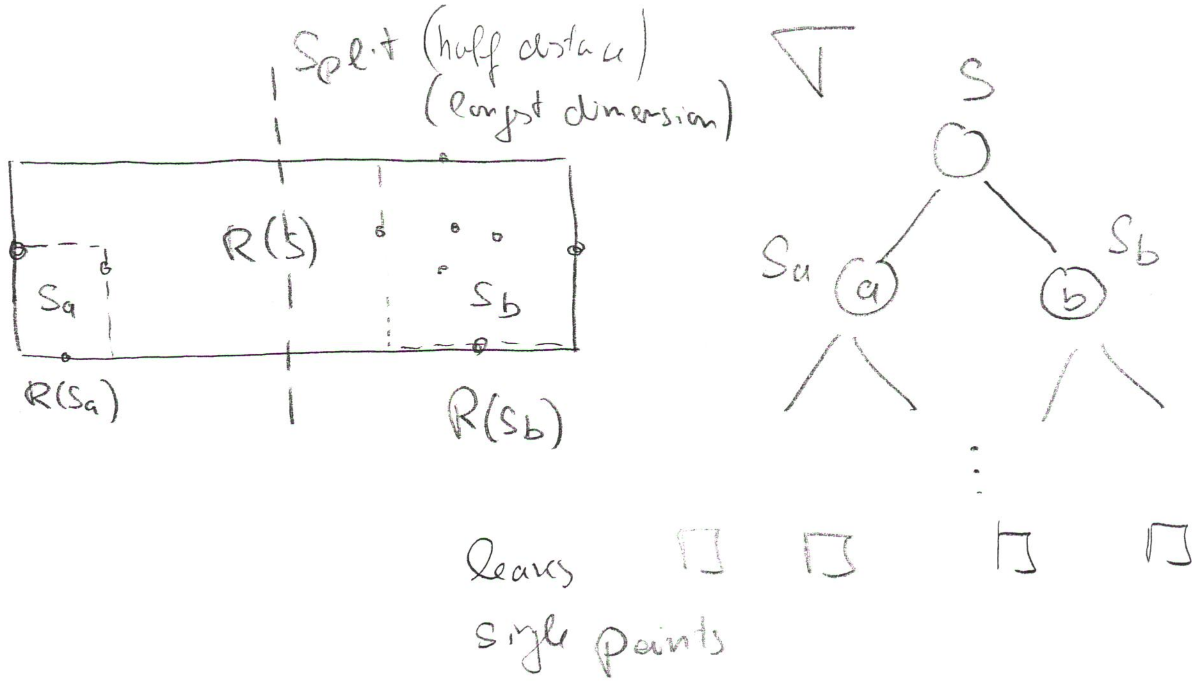
Proof of Theorem 19

① Algorithm for WSPD of S

② Analysis of Alg.

Start with ① Split-Tree construction in $O(dn \log n)$

- Start with bounding box $R(S)$
- halve the longest dimension of $R(S)$ by orthogonal hyperplane
- recur. on resulting subsets S_a S_b of S until points are left



Split tree construction using sorted lists in any dimension!

- Independent from s
- nodes $a \to$ subset S_a of S
- $O(dn \log n)$ construction time (partial split trees)

<u>Construction:</u> (with partial split trees)

avoids resorting:

- Passing down sorted lists of coordinates (doubly linked sorted lists) for each dimension (for boundary box computation)
- Split the list in time proportional to smallest subset ~~by~~ pointers moving from each end until split is found

($\overset{\rightarrow}{\cdot}$ . . . . $\overset{\leftarrow}{\cdot}$) [e.g. $d=1$] $\rightsquigarrow$

- Recursively constructing partial split trees containing $\leq \frac{n}{2}$ points in each leaf (one phase)
- Compute the partial split trees for the leaves? ($\leq \frac{n}{2}$ points)

Overall $O(d\, n \log n)$ time

One partial split tree: $d \cdot O(n) + d \cdot \sum_{u \text{ is leaf}} |S_u| \in O(n)$ $\quad |S_u| \leq \frac{n}{2}$

Running time:

$$T(n) = d \cdot O(n) + d \cdot \sum_u T(|S_u|)$$

$$\in O(n \log n) \qquad \dot{\bigcup_u} S_u = S$$

$|S_u| \leq \frac{n}{2}$

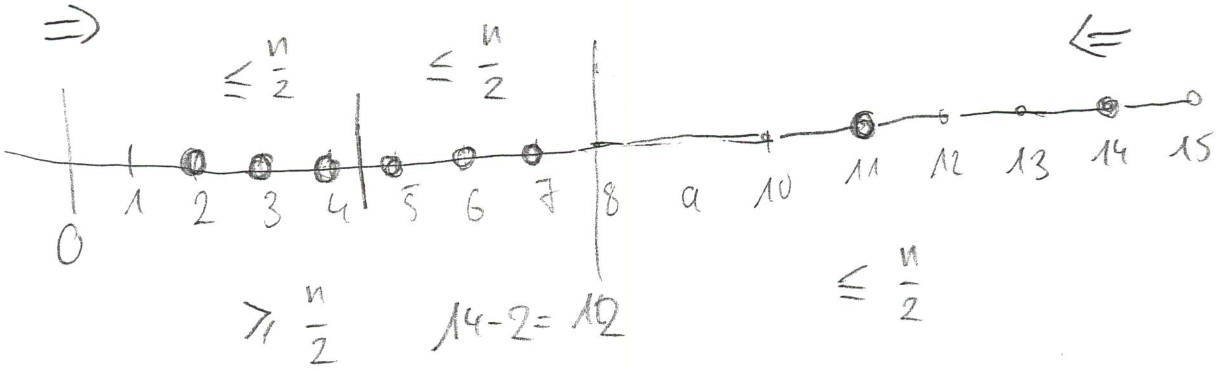
$$T(n) = d\, O(n) + d \sum_{u \in L_1} T(|S_u|)$$

$$= d \cdot O(n) + d \cdot \sum_{i=1}^{k} O(n_i) + d \cdot \sum_{u \in L_2} T(S_u) \qquad |S_u| \leq \frac{n}{4}$$

$\vdots$

$$= d \cdot \log n \; O(n) + d \cdot \sum_{u \in L_{\log n}} T(|S_u|) \qquad |S_u| \leq 1$$

Example (One phase)



Worst-case