Brunn's Inequality

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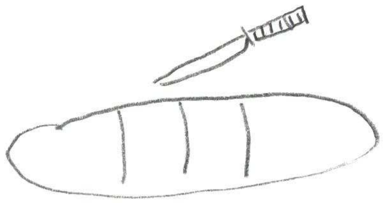
Chapter 12.2

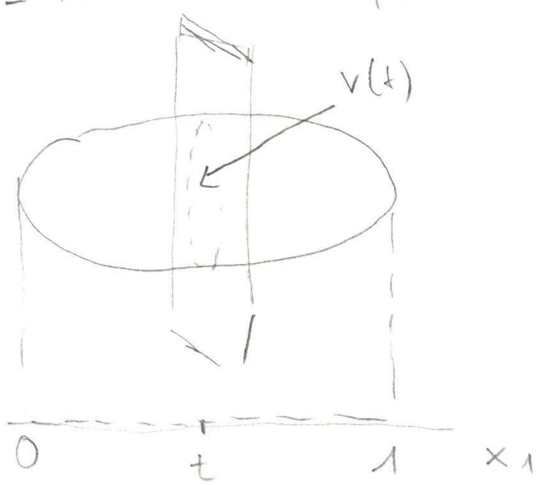
Problem: Bread slices taken from a bread!



Take the middle one because it is never the smallest!

This holds in any dimension!

Situation:
$C \subseteq \mathbb{R}^{d+1}$ convex, $[t_{min}, t_{max}] = [0,1]$ w.l.o.g.

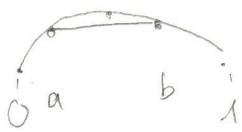


Projection of C onto x_1 - axis

$v(t) :=$ Volume of $C \cap$ (Hyperplane) $\{x_1 = t\}$

Theorem 32: (Brunn's inequality):
The function $v(t)^{\frac{1}{d}}$ is concave over $[0,1]$.

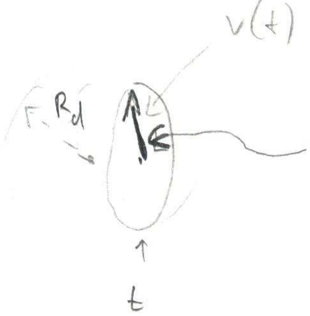
$f(t)$ concave over $[0,1]$



$:\Longleftrightarrow \forall a,b \in [0,1] : (1-t) f(a) + t f(b) \leq f(t) \quad \forall t \in [a,b]$

$v(t)^{\frac{1}{d}}$?

Intuition: d-dimensional sphere of volume $\underline{\underline{1}}$
with radius R_d



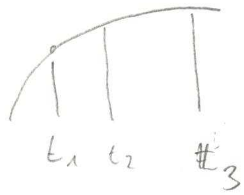
$\underline{\underline{R_d \cdot v(t)^{\frac{1}{d}}}}$ Radius of sphere with volume $v(t)$

Brunn's inequality implies Dread-solution:

$v(t)^{\frac{1}{d}}$ concave $\forall \; 0 \le t_1 \le t_2 \le t_3 \le 1$

$$v(t_2)^{\frac{1}{d}} \geq \min\left(v(t_1)^{\frac{1}{d}}, v(t_3)^{\frac{1}{d}}\right)$$

$$\Rightarrow v(t_2) \geq \min\left(v(t_1), v(t_3)\right)$$



[$v(t)$ is not concave in general]
$d \geq 3$

(Compact { [limit points inside] } $\Leftrightarrow$ closed and bounded [ball of finite radius])

Proof of Theorem 32 makes use of

Theorem 33 (Brunn-Minkowski-Inequality)

Let $A, B \subseteq \mathbb{R}^d$ compact non-empty:

Then $vol(A)^{\frac{1}{d}} + vol(B)^{\frac{1}{d}} \leq vol(A+B)^{\frac{1}{d}}$

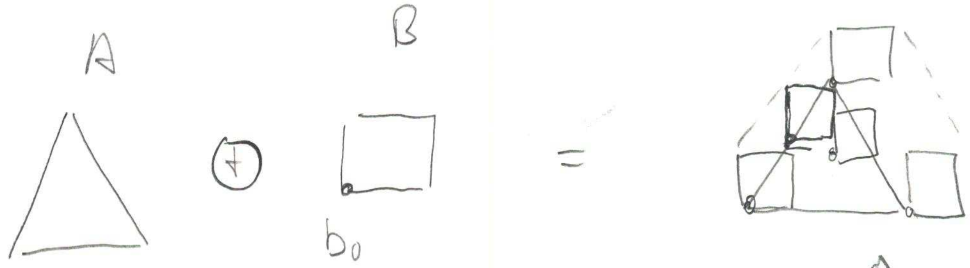
Here "$\oplus$" Minkowski-Sum

$$A \oplus B := \{a+b \mid a \in A, b \in B\}$$

A' translation of A
B' translation of B $\Rightarrow$ $A' \oplus B'$ translation of $A \oplus B$

Position of $A \oplus B$ depends on the choice of the coordinate system. [volumes independent from translations]

Interpretation:



Fix A, pick b_0 in B, translate B into all situations where b_0 is in A
(Result modulo translation)

Proof of Theorem 32 with Theorem 33:

Let $A, B \subseteq \mathbb{R}^d$ be convex. For $t \in [0,1]$

Consider $Z_t := (1-t)A \oplus tB = \{(1-t)a + tb \mid a \in A, b \in B\}$

"Morphing of A into B" (A, B will be the outer cuts)

Realize morphing geometrically

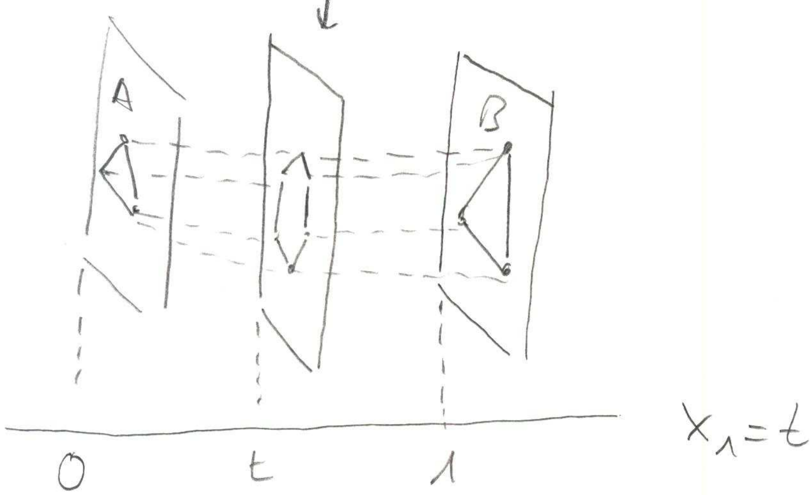
Place A and B into $\mathbb{R}^{d+1}$

A into hyperplane $\{x_1 = 0\}$ Line segments between them?

B into hyperplane $\{x_1 = 1\}$

Situation:

Approx. of $v(t)^{\frac{1}{d}}$



Lemma 34: A, B convex

$$\operatorname{conv}\big((\{0\} \times A) \cup (\{1\} \times B)\big) \overset{!}{=} \bigcup_{t \in [0,1]} \{t\} \times \underbrace{\big((1-t)A + tB\big)}_{Z_t} =: C'$$

Proof: "$\supseteq$" Clearly C' contains

$\{0\} \times A$ and $\{1\} \times B$ and

all lines from A to B

Any convex set that contains $\{0\} \times A$ and $\{1\} \times B$

also contains C'

" $\subseteq$ " Clearly: $\{0\}\times A$ and $\{1\}\times B \subseteq C'$

for $t = 1, 0$.

C' is convex $\left(\Rightarrow \quad \text{conv}(\dots) \subseteq C' \right)$ $\underset{C' \subseteq A}{\frown}$

$p = (t_1, (1-t_1)a_1 + t_1 b_1) \quad q = (t_2, (1-t_2)a_2 + t_2 b_2) \in C'$

$(a_1, a_2 \in A, b_1, b_2 \in B)$

Line segment between p and q:

$r \in [0,1] \quad (1-r)p + rq =$

$$\Big((1-r)t_1 + r t_2 , (1-r)\big((1-t_1)a_1 + t_1 b_1\big) + r\big((1-t_2)a_2 + t_2 b_2\big) \Big)$$

$$= \Big((1-r)t_1 + r t_2 , \big(1 - \underbrace{((1-r)t_1 + r t_2)}_{\delta}\big) \underbrace{\big((1-\lambda)a_1 + \lambda a_2\big)}_{\in A \text{ (convex)}}$$

A, B convex

$$+ \underbrace{\big((1-r)t_1 + r t_2\big)}_{\delta} \underbrace{\big((1-\mu)b_1 + \mu b_2\big)}_{\in B \text{ (convex)}} \Big) \overset{!}{\in} C$$

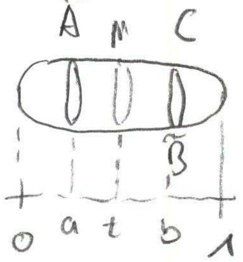
where $\lambda := \dfrac{r(1-t_2)}{1-\delta} \qquad \mu := \dfrac{r t_2}{\delta}$ (verify!)

$\in [t_1, t_2]$

(Simple computations) $\square$

Now Proof of Theorem 32:

Let $C \subseteq \mathbb{R}^{d+1}$ convex with projection $[0,1]$ onto x_1-axis



Let A and B be two "cuts"

Let $\tilde{A} = C \cap \{x_1 = a\}$, $\tilde{B} = C \cap \{x_1 = b\}$ $\quad \tilde{A}, \tilde{B} \in \mathbb{R}^{d+1}$

Volumes (d-dim) $v(a)$, $v(b)$

Show: $t \in [a,b]$ ($v(t)^{\frac{1}{d}}$ is concave)

$$(1-t)\, \underbrace{v(a)}_{\mathrm{vol}(\tilde{A})}{}^{\frac{1}{d}} + t\, \underbrace{v(b)}_{\mathrm{vol}(\tilde{B})}{}^{\frac{1}{d}} \leq \underbrace{v(t)}_{\mathrm{vol}(M)}{}^{\frac{1}{d}}$$

W.l.o.g. $a = 0$, $b = 1$

Instead of C and M consider $C' := \mathrm{conv}(\tilde{A} \cup \tilde{B})$, $M' = C' \cap \{x_1 = t\}$

Clearly: $C' \subseteq C$, $M' \subseteq M$ (C convex)

(Lemma 34) $\Rightarrow$ $M' = (1-t)\tilde{A} + t\tilde{B}$ (volume is d-dim.)

$$\Rightarrow (1-t)\,\mathrm{vol}(\tilde{A})^{\frac{1}{d}} + t\,\mathrm{vol}(\tilde{A}) \overset{\text{triv. (scaling)}}{=} \mathrm{vol}((1-t)\tilde{A})^{\frac{1}{d}} + \mathrm{vol}(t\tilde{B})^{\frac{1}{d}}$$

$$\underset{\text{Theorem 33}}{\leq} \mathrm{vol}((1-t)\tilde{A} + t\tilde{B})^{\frac{1}{d}} = \mathrm{vol}(M')^{\frac{1}{d}}$$

$$\underset{M' \subseteq M}{\leq} \mathrm{vol}(M)^{\frac{1}{d}} \qquad \square$$

Now the more general result:

Proof: (Brunn-Minkowski)

First A, B union of finitely many closed axis-parallel boxes ("bricks") with disjoint interior

Show the proof for the union of k such boxes
(Afterwards arbitrary A, B approximated)

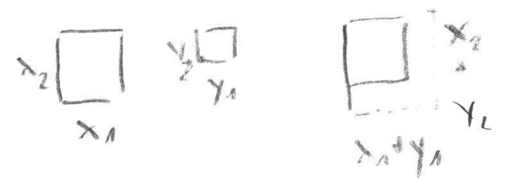
Ind. over k

$k=2$ (non-empty) A and B are bricks and $A \oplus B$ is a brick also

A given by side length $x_1, x_2, \dots, x_d$
B given by " " $y_1, y_2, \dots, y_d$

Show: $\left(\prod_{i=1}^{d} x_i\right)^{1/d} + \left(\prod_{i=1}^{d} y_i\right)^{\frac{1}{d}} \le \left(\prod_{i=1}^{d} (x_i + y_i)\right)^{\frac{1}{d}}$

Holds: example $d=2$



$\sqrt{x_1x_2} + \sqrt{y_1y_2} \le \sqrt{(x_1+y_1)(x_2+y_2)}$ $\quad ( )^2$

$\Longleftrightarrow \left[\left(\sqrt{x_1}\sqrt{x_2} + \sqrt{y_1}\sqrt{y_2}\right)^2 \le (x_1+y_1)\cdot(x_2+y_2) \right]$

$\cancel{x_1x_2} + \cancel{y_1y_2} + 2\sqrt{(x_1x_2)(y_1y_2)} \overset{!}{\le} \cancel{x_1x_2} + \cancel{y_1y_2} + y_1x_2 + y_2x_1$

$$\boxed{\begin{aligned} 2\sqrt{AB} &\overset{!}{\le} AB \\ 4\,AB &\le (A+B)^2 \\ &= A^2 + 2AB + B^2 \\ 0 &\le (A-B)^2 \end{aligned}}$$

$d \geq 2 \quad r = 2$

Lemma 35 $x_1, \dots, x_d, y_1, \dots, y_d \geq 0$

Then $\left(\prod_{i=1}^{d} x_i\right)^{\frac{1}{d}} + \left(\prod_{i=1}^{d} y_i\right)^{\frac{1}{d}} \leq \left(\prod_{i=1}^{d} (x_i + y_i)\right)^{\frac{1}{d}}$

Proof: Let $x_i^{\frac{1}{d}} =: v_i$, $y_i^{\frac{1}{d}} =: w_i$ and apply $(\ )^d$

$$\left(\prod_{i=1}^{d} v_i + \prod_{i=1}^{d} w_i\right)^d \overset{!}{\leq} \prod_{i=1}^{d} (v_i^d + w_i^d)$$

$\Uparrow$

$\left[(x_1 x_2)^{\frac{1}{d}} = x_1^{\frac{1}{d}} x_2^{\frac{1}{d}} \right]$

Binomial-coefficients

$$\Leftrightarrow \underbrace{\sum_{\mu=0}^{d} \binom{d}{\mu} (v_1 \cdots v_d)^{\mu} (w_1 \cdots w_d)^{d-\mu}}_{l_\mu} \leq \sum_{\mu=0}^{d} \overbrace{\sum v_{i_1}^d \cdots v_{i_\mu}^d w_{j_1}^d \cdots w_{j_\mu}^d}^{r_\mu}$$

$\binom{d}{\mu}$ Summands

$i_1, \dots, i_\mu \in \{1, \dots, d\} \quad i_m \neq i_n$

$j_1, \dots, j_{\mu-d} \in \{1, \dots, d\} \quad j_m \neq j_n$

Show: $0 \leq \mu \leq d \qquad \dfrac{r_\mu}{\binom{d}{\mu}} \geq \dfrac{l_\mu}{\binom{d}{\mu}}$ $\qquad \left[\binom{d}{\mu} = \dfrac{d!}{\mu!(d-\mu)!} \right]$

$\ulcorner$ μ fix $\lrcorner$

$$\frac{r_\mu}{\binom{d}{\mu}} = \frac{1}{\binom{d}{\mu}} \sum_{\substack{\{i_1,\dots,i_\mu\}\subseteq\{1,\dots,d\}\ i_m\neq i_n \\ \{j_1,\dots,j_{d-\mu}\}\subseteq\{1,\dots,d\}\ j_m\neq j_n}} v_{i_1}^d \cdots v_{i_\mu}^d\, w_{j_1}^d \cdots w_{j_{d-\mu}}^d$$

$\ulcorner$ # $\binom{d}{\mu}$ summands $\lrcorner$

$$\geq \Bigg(\prod_{\substack{\{i_1,\dots,i_\mu\}\subseteq\{1,\dots,d\}\ i_m\neq i_n \\ \{j_1,\dots,j_{d-\mu}\}\subseteq\{1,\dots,d\}\ j_m\neq j_n}} v_{i_1}^d \cdots v_{i_\mu}^d\, w_{j_1}^d \cdots w_{j_{d-\mu}}^d \Bigg)^{\frac{1}{\binom{d}{\mu}}}$$

arith $\geq$ geom

$\ulcorner$ $\frac{x_1+\dots+x_n}{n} \geq (x_1 x_2 \cdots x_n)^{\frac{1}{n}}$ $\lrcorner$

$$\ulcorner\ \frac{d\binom{d-1}{\mu-1}}{\binom{d}{\mu}} = \frac{\cancel{(d-\mu)!}\,\mu!\,d\cancel{(d-1)!}}{\cancel{d!}\,\cancel{(d-1-\mu+1)!}\,(\mu-1)!} = \mu\ \lrcorner$$

$$= \frac{l_\mu}{\binom{d}{\mu}}$$

each v_i occurs $\binom{d-1}{\mu-1}$ times!

$\ulcorner$ $\binom{d-1}{\mu-1}$ subsets of $\{1,\dots,d\}\setminus\{i\}$ can be enlarged by v_i $\lrcorner$

each w_j occurs $\binom{d-1}{d-\mu-1}$ times

$\ulcorner$ $\binom{d-1}{d-\mu-1}$ subsets of $\{1,\dots,d\}\setminus\{j\}$ can be enlarged by w_j $\lrcorner$

$$\Rightarrow\quad v_i^{\,d\binom{d-1}{\mu-1}\frac{1}{\binom{d}{\mu}}} = v_i^{\mu}$$

$$w_j^{\,d\binom{d-1}{d-\mu-1}\frac{1}{\binom{d}{\mu}}} = w_j^{\,d-\mu} \quad \rightsquigarrow (v_1\cdots v_d)^{\mu}(w_1\cdots w_d)^{d-\mu}$$

$\square$