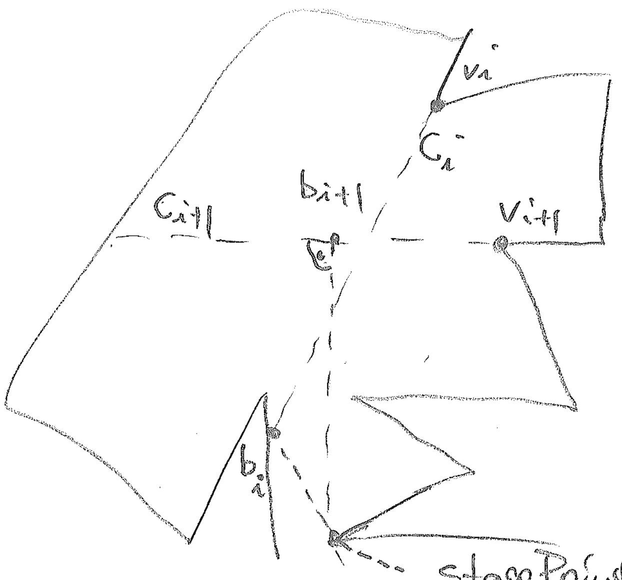
Analysis of group exploration

Lemma 3 BasePoints $b_1, b_2, \ldots, b_m$ generated by calls to ExploreRightVertex, within one execution of ExploreRightGroup, are in clockwise order around StagePoint

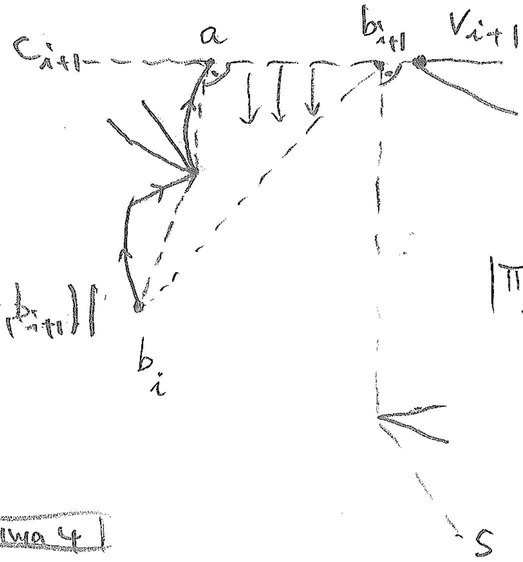
Proof Let b_i, b_{i+1} be on cuts of right vertices v_i, v_{i+1}. Since ExploreRightVertex proceeds clockwise – first, v_i comes before v_{i+1} in clockwise order on ∂P:



b_i below cut c_{i+1}, because v_{i+1} explored after

StagePoint, or last vertex on shortest paths from StagePoint to b_i, b_{i+1} must be below both cuts c_i, c_{i+1}

Lemma 4 Robot's path between consecutive BasePoints is at most 3 times as long as the shortest path



$|ab_{i+1}| \leq$
$\leq |\text{shortestpath}(b_i, b_{i+1})|$
by projection.

$$|\Pi_{b_i}^{b_{i+1}}| = |\Pi_{b_i}^{a}| + |ab_{i+1}|$$

$$|\Pi_{b_i}^{a}| \underset{\text{Lemma 2}}{\leq} 2|\text{Shortestpath}(b_i, a)|$$

$$\leq 2|\text{Shortestpath}(b_i, b_{i+1})|$$

because a = closest point to b_i on c_{i+1}

Lemma 5 For path π generated by one call of ExploreRightGroup, we have

$$|\pi| \leq 3\sqrt{2} \cdot |RCH(\{b_1, \ldots, b_m\})|,$$

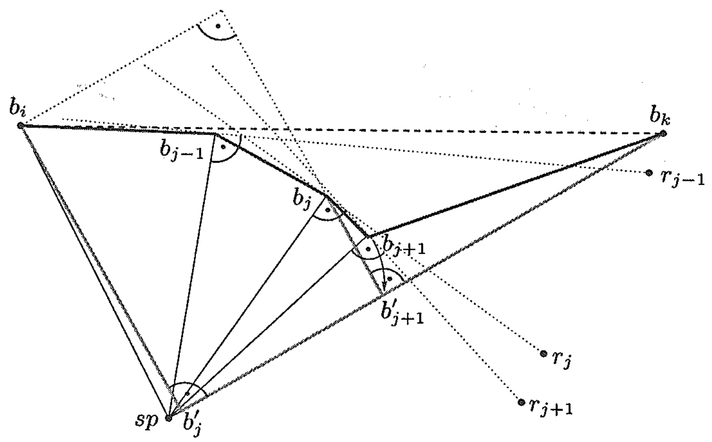
where $b_1, \ldots, b_m$ are the BasePoints generated.

Proof If each b_i were vertex of RCH, even factor 3 would hold, by Lemma 4. Otherwise, assume $b_i, b_k \in$ RCH but $b_{i+1}, \ldots, b_{k-1}$ are not; in particular, they are $\notin \partial P$.

Cut of r_j must pass above BasePoints $b_i, \ldots, b_{j-1}$, by order of exploration.

Shortest paths from StagePoint sp hit cuts of BasePoints $\notin \partial P$ at 90° angle.

The following operation is repeated backwards for all BasePoints like b_{j+1} where a left turn occurs.



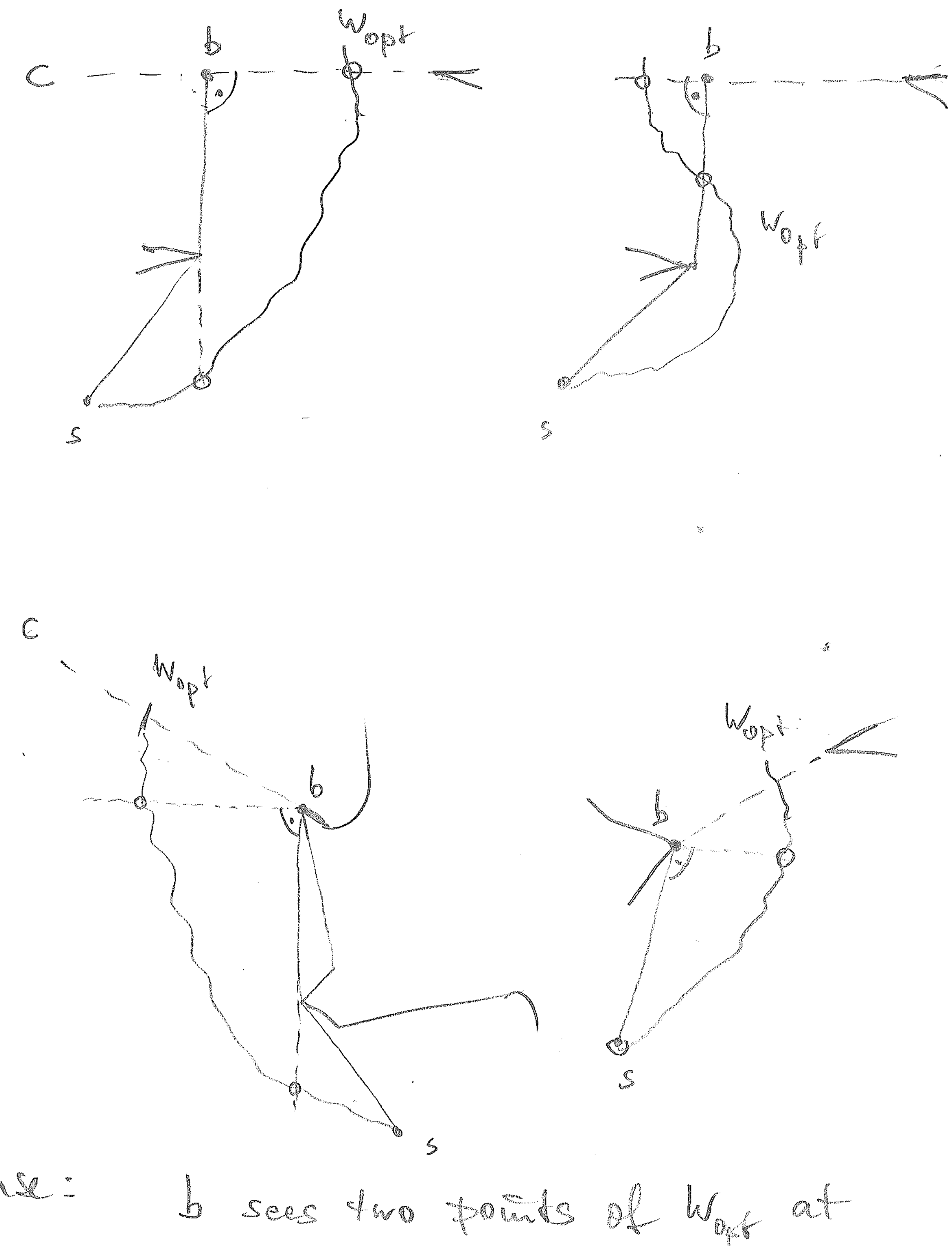
For analysis, move b_{j+1} to b'_{j+1}, and replace convex chain $b_i, b_{j-1}, \ldots, b_j, b'_{j+1}$ by 90° wedge $b_i - b'_j - b'_{j+1}$ of length $\leq \sqrt{2} \cdot |b_i b_k|$.

$\boxed{\text{Lemma 5}}$

Crucial: <u>Lemma 6</u> All BasePoints are contained in angle hull $AH(W_{opt})$.

<u>Proof</u> Let b be a BasePoint $\overset{}{\underset{\text{Def.}}{\Longrightarrow}}$ b is closest point to s on cut C:

W_{opt} must visit cut C, too, and it comes from s



in each case: b sees two points of W_{opt} at $90°$ angle

(that's why BasePoints need adjustment!)

Next, group exploration becomes recursive.

procedure *ExploreRightGroupRec* (**in** *TargetList*);
  ExploreRightGroup (*TargetList*, *ToDoList*);   (* *ToDoList* gets filled in. *)
  Clean up *ToDoList*:
    retain only those right vertices in *ToDoList*
    which are highest up in the shortest path tree;
  for all vertices v of *ToDoList* in clockwise order **do**
    walk on the shortest path to v;   (* connect stage points *)
    ExploreLeftGroupRec({all known left descendants of v in counterclw. order});
end *ExploreRightGroupRec*;

Explores, from CP, all (right) vertices in TargetList and everything below them in shortest path tree

After first ExploreRightGroup, ToDoList contains all purely right descendants of current Stage Point CP, that have left children. Only the topmost ones need to be kept, with links to their known left descendants.

These are visited in <u>clockwise</u> order, and explored by symmetric Explore<u>Left</u>Group recursively.

Finally, the topmost procedure of our exploration algorithm:

procedure *ExplorePolygon* (**in** P, **in** s);
  ExploreRightGroup ({clockwise list of all right vertices visible from s}, *ToDoList*);
  TargetList := {all left children of the vertices of *ToDoList*};
  Add all left vertices visible from s into *TargetList* and sort counterclockwise;
  ExploreLeftGroupRec (*TargetList*);
end *ExplorePolygon*;

So much for the strategy.
Some analysis remains to be done.

basic procedures: Explore $\left\{ \begin{matrix} \text{Left} \\ \text{Right} \end{matrix} \right\}$ Group

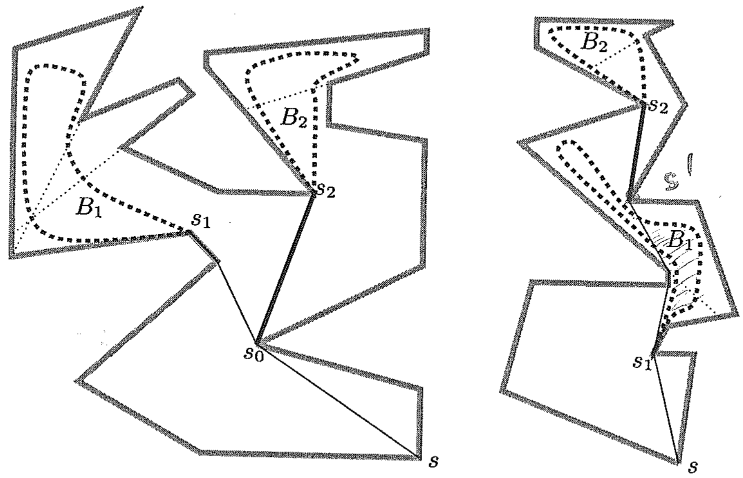
both generate sets of BasePoints and return to their StagePoints.

define recursively level of BasePoints generated:
first call to Explore Right Group: level 0
then level = level+1 in recursion

read levels mod 3 : three categories, 0, 1, 2

<u>Lemma 7</u> Let <u>BasePoint</u> sets B_1 B_2 be of the same category. Then RCH(B_1), RCH(B_2) are mutually invisible, except, (possibly) for their StagePoints.

<u>Proof:</u>



B_1, B_2 : same level

$\geq$ three levels apart
no point of B_2 can see shortest path from s_2 to its predecessor, s'.

Consequence: <u>Lemma 8</u> :

[Lemma 7]

$$| \mathrm{RCH}(B_1) | + | \mathrm{RCH}(B_2) | \leq | \mathrm{RCH}(B_1 \cup B_2) |$$

perimeter

[Lemma 8]

Now we can prove the main result.

Theorem B Let P be a simple polygon and $s \in \partial P$. Then strategy ExplorePolygon produces an exploration tour π through s satisfying

$$|\pi| \leq \underbrace{(18\sqrt{2}+1)}_{\leq 26.5} |W_{opt}(s)|$$

Proof Fix one of the three categories of BasePointSet, and let $B_1, \dots, B_m$ be the BasePoint sets contained therein. Each one is produced by a closed subtour in Explore$\left\{\begin{smallmatrix}\text{Right}\\ \text{Left}\end{smallmatrix}\right\}$Group.

Let $L :=$ total path length spent on this category.

$$L \underset{\text{Lemma 5}}{\leq} \sum_{i=1}^{m} 3\sqrt{2}\, |RCH(B_i)| \underset{\text{Lemma 8}}{\leq} 3\sqrt{2}\, \left|RCH\left(\bigcup_{i=1}^{m} B_i\right)\right|$$

$$\underset{\substack{\text{Lemma 6}\\ \text{minimality } RCH(\bigcup B_i)}}{\leq} 3\sqrt{2}\, |RCH(AH(W_{opt}))|$$

$$\underset{\text{clear}}{\leq} 3\sqrt{2}\, |AH(W_{opt})|$$

$$\underset{\text{Theorem A}}{\leq} 3\sqrt{2} \cdot 2 \cdot |W_{opt}|$$

$\Rightarrow$ path length of all round trips in all 3 categories

$$\leq 18 \cdot \sqrt{2}\, |W_{opt}|$$

Still missing: Round trips in Explore$\left\{\begin{smallmatrix}\text{Right}\\ \text{Left}\end{smallmatrix}\right\}$GroupRec used for connecting StagePoints ✓

- these StagePoints are visited by π in clockwise order around ∂P
- W_{opt} must visit these points, too
- W_{opt} cannot cross itself, by minimality

$\Rightarrow$ W_{opt} visits StagePoints in clockwise order, too

$\Rightarrow$ $|\text{All round trips}| \leq |W_{opt}|$

$\boxed{\text{Theorem B}}$

(26.5 = best bound since '01, but probably way too large

Remains to prove <u>Theorem A</u> (just used in Proof of Theorem B:)

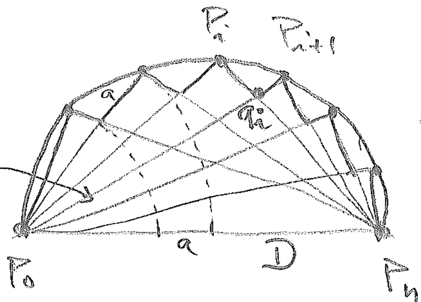
$$|RCH(D)| \leq 2 \cdot |D| \qquad \text{(wlog } D \text{ relatively convex)}$$

and this bound can be attained.

First, we show that this factor of 2 can be attained:

D = segment of length 1, P = jagged circle

every angle $= \frac{\pi}{2n}$



total length of all ascending segments: ≤ 1 (by projection)

same for descending segments

$P = AH(D)$ (only upper half shown)

How much is missing?

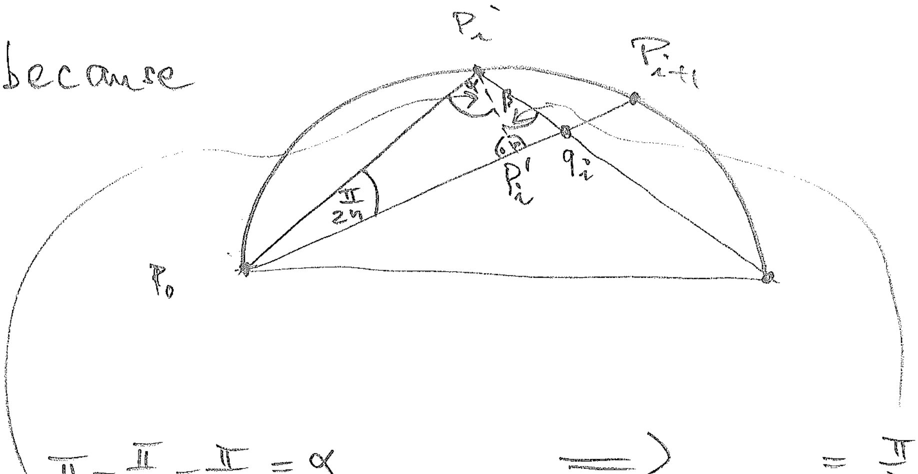
Let $p_i' :=$ orthogonal projection of p_i onto $\overline{p_0 p_{i+1}}$

missing locally: $|p_0 q_i| - |p_0 p_i| \underset{\uparrow}{\leq} |p_0 q_i| - |p_0 p_i'| = |p_i' q_i|$

$|p_0 p_i'| \leq |p_0 p_i|$

$= |p_i q_i| \sin$

because



$\pi - \frac{\pi}{2} - \frac{\pi}{2n} = \alpha \quad \Longrightarrow \quad \underset{\text{Thales}}{=} \frac{\pi}{2} - \alpha = \frac{\pi}{2n} =: \beta$

$\Rightarrow |p_i' q_i| = |p_i q_i| \sin\beta = |p_i q_i| \sin\frac{\pi}{2n}$

sum of all distances missing from 1:

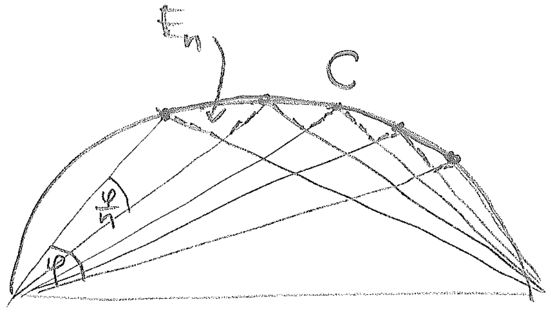
$$\sum_i \left(|p_0 q_i| - |p_0 p_i| \right) \leq \sum_i |p_i q_i| \sin\frac{\pi}{2n}$$

$$= \underbrace{\sin\frac{\pi}{2n}}_{\to 0 \text{ as } n\to\infty} \underbrace{\sum_i |p_i q_i|}_{\text{total length of all descending segments } \leq 1}$$

$\Rightarrow$ factor 2 can be realized.

Now, to the (more important) upper bound.

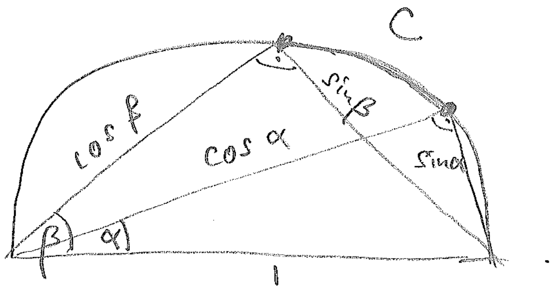
Based on the previous construction, we define for each circular arc C its jagged length, $J(C)$, as follows:



$$J(C) = \overline{\lim_{n\to\infty}} |E_n|$$

supremum value of lengths of ⋀⋀⋀ generated by partitioning φ into n equal ang

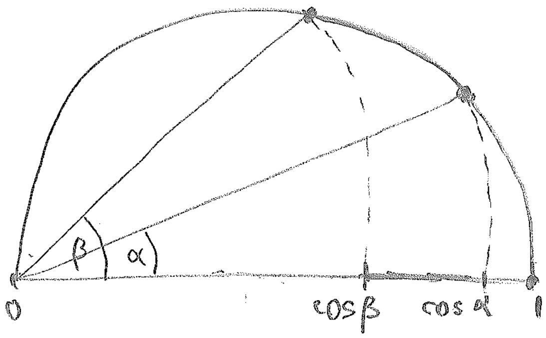
<u>Lemma 9</u> Let C be defined by



Then,

$$J(C) = \sin\beta - \sin\alpha - \cos\beta + \cos\alpha .$$

<u>Proof</u> As above, proving that error in adding up rotated images of ascending and descending segments tends to 0.



sum of ascending piec $\sim \cos\alpha - \cos\beta$

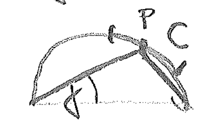
analogously for descending piece

[Lemma 9]

We can write

$$J(C) = \sin\beta - \sin\alpha - \cos\beta + \cos\alpha = \int_\alpha^\beta (\cos\gamma + \sin\gamma)\, d\gamma$$

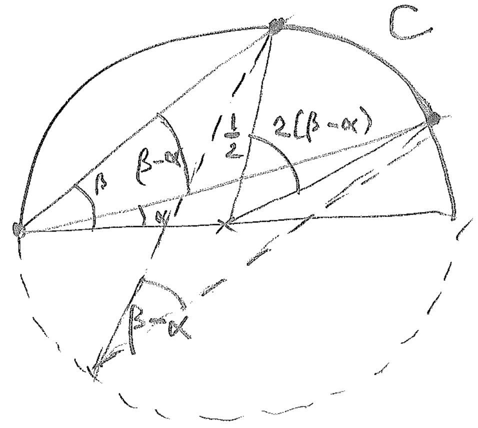
(*) $$= \int_0^{\pi/2} C_\gamma \, d_\gamma , \quad \text{where} \quad C_\gamma = \begin{cases} \text{length of} \\ \text{if apex } p \in C \\ 0, \text{ otherwise} \end{cases}$$



(also true for radii $\neq 1$)

Lemma 10 $\quad J(C) \geq |C|$

Proof Assume $C \subset$ circle of diameter 1 (scaling!)



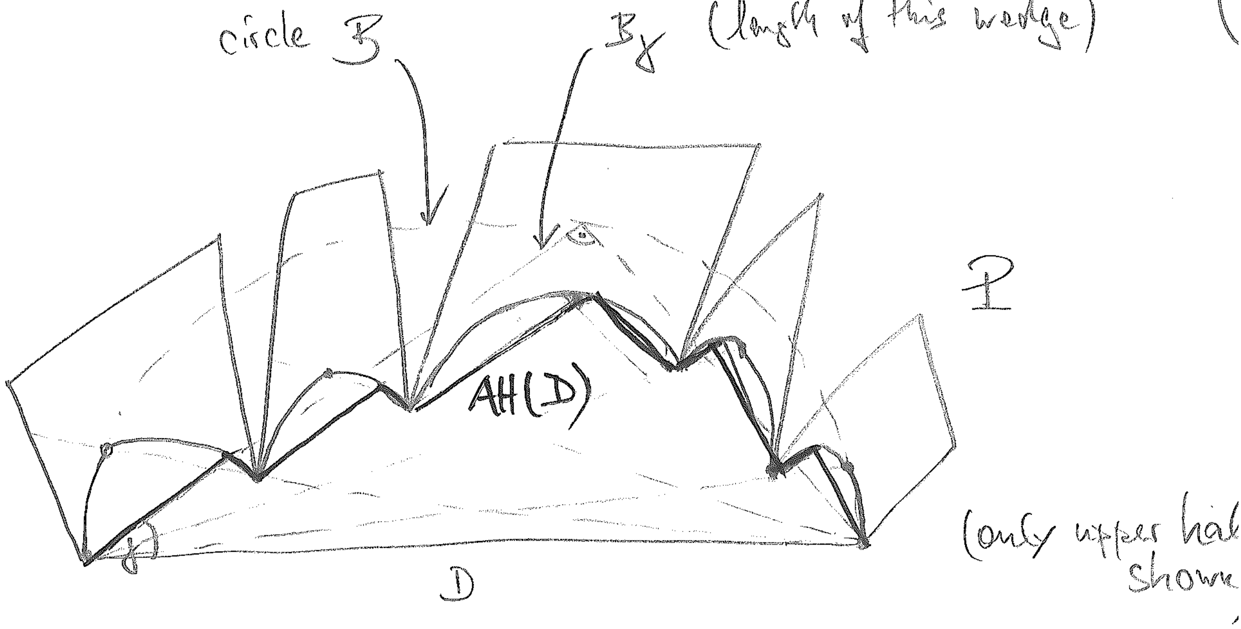
$$J(C) = \int_\alpha^\beta \underbrace{(\cos\gamma + \sin\gamma)}_{\geq 1} d\gamma$$

$$\geq \int_\alpha^\beta 1 \, d\gamma = \beta -$$

$$= |C$$

$\boxed{\text{Lemma 10}}$

Now let D be a line segment inside polygon P such that Att(D) touches P only at vertices of P (i.e., photographer need not follow a wall).



$$
\left|\widehat{AH}(D)\right| \underset{\text{Lemma 10}}{\leq} J\left(\widehat{AH}(D)\right) = \sum_{C \in AH(D)} J(C) \overset{(*)}{=} \sum_{C \in AH(D)} \int_0^{\frac{\pi}{2}} C_\gamma \, d\gamma = \int_0^{\frac{\pi}{2}} \Big(\sum_{C \in AH(D)} C_\gamma \Big) \, d\gamma = \int_0^{\frac{\pi}{2}} B_\gamma \, d\gamma = J(B) = |D|
$$

upper part of angle hull; same goes for the lower one.

Can be generalized to arbitrary D.

$\boxed{\text{Theorem A}}$