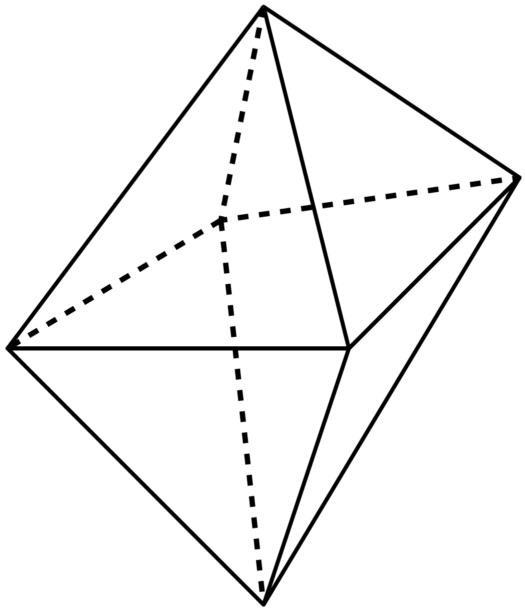
Convex Polytope (Chapter 5.1 and 5.2)



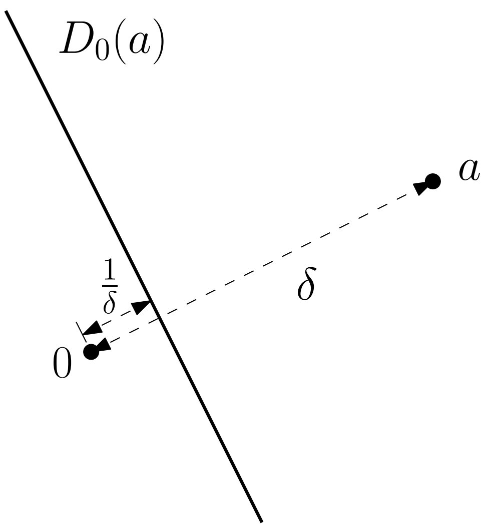
A convex polytope is a convex hull of finite points in $\mathbb{R}^d$

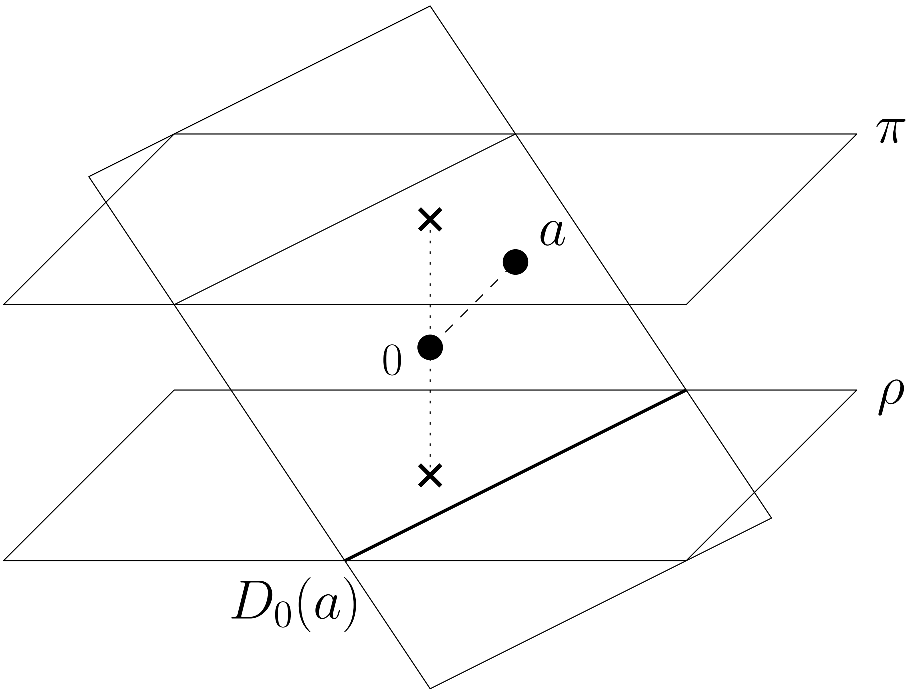
- bounded convex polyhedron

Central Geometric Duality D_0
For a point $a \in \mathbb{R}^d \setminus \{0\}$, it assigns the hyperplane

$$D_0(a) = \{x \in \mathbb{R}^d \mid \langle a, x \rangle = 1\},$$

and for a hyperplane h not passing through the origin, where $h = \{x \in \mathbb{R}^d \mid \langle a, x \rangle = 1\}$, it assisns the points $D_0(h) = a \in \mathbb{R}^d \setminus \{0\}$.



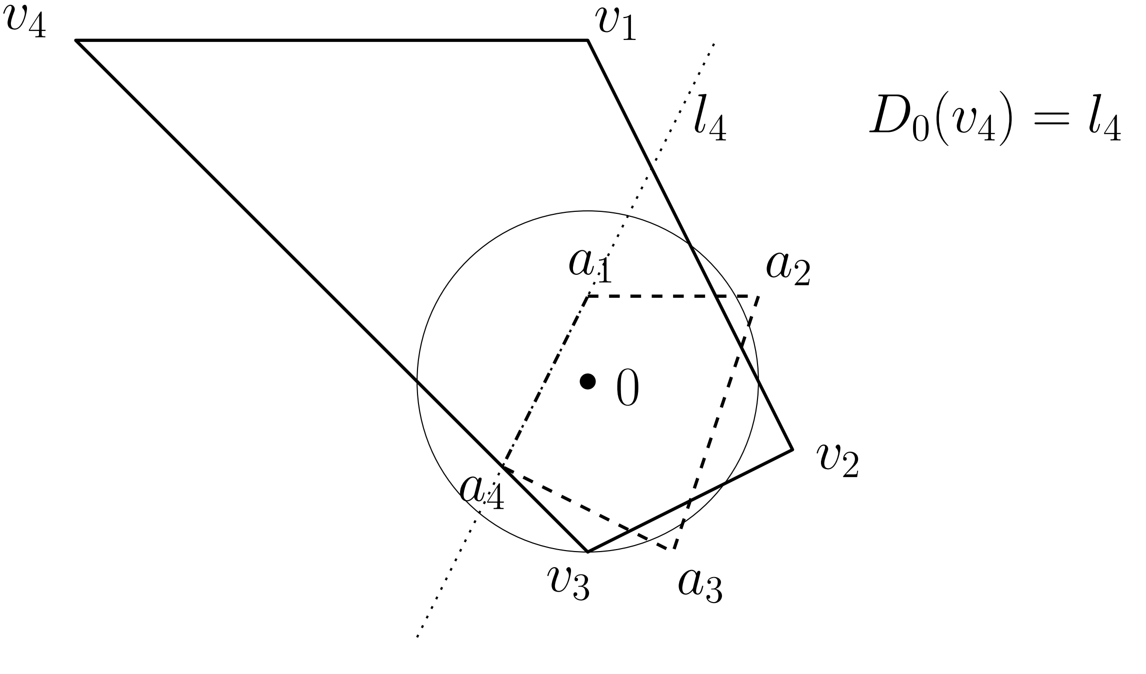


An interpretation of duality through $\mathbb{R}^{d+1}$

- "Primal" $\mathbb{R}^d$: the hyperplane $\pi = \{x \in \mathbb{R}^{d+1} \mid x_{d+1} = 1\}$
- "dual" $\mathbb{R}^d$: the hyperplane $\rho = \{x \in \mathbb{R}^{d+1} \mid x_{d+1} = -1\}$
- A point $a \in \pi$
  - construct the hyperplane in $\mathbb{R}^{d+1}$ perpendicular to $0a$ and containing 0
  - intersect the hyperplane with ρ

k-flat is a hyperplane in $(k+1)$ dimensions.

- 0-flat is a point, 1-flat is a line, 2-flat is a plane, and so on.
- The dual of a k-flat is a $(d-k-1)$-flat.



Half-space

For a hyperplane h not containing the origin, let h^- stand for the closed half-space bounded by h and containing the origin, while h^+ denotes the other closed half-space bounded by h. That is, if $h = \{x \in \mathbb{R}^d \mid \langle a, x \rangle = 1\}$, then $h^- = \{x \in \mathbb{R}^d \mid \langle a, x \rangle \leq 1\}$ and $h^+ = \{x \in \mathbb{R}^d \mid \langle a, x \rangle \geq 1\}$.

Duality preserves incidences

For a point $p \in \mathbb{R}^d \setminus 0$ and a hyperplane h not containing the origin,

- $p \in h$ if and only if $D_0(h) \in D_0(p)$.
- $p \in h^-$ if and only if $D_0(h) \in D_0(p)^-$.
- $p \in h^+$ if and only if $D_0(h) \in D_0(p)^+$.

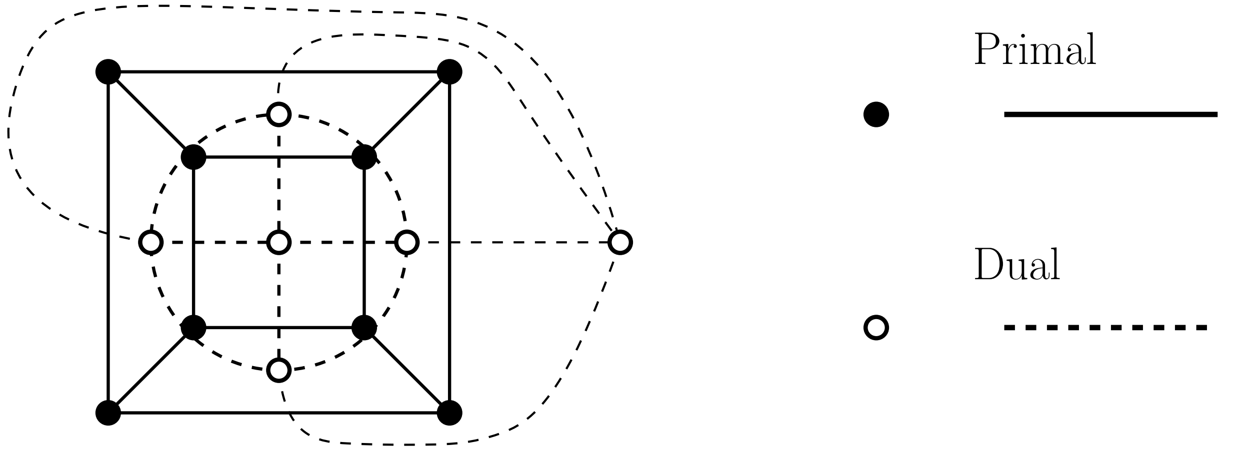
Dual set (Polar set)

For a set $X \subseteq \mathbb{R}^d$, the set dual to X, denoted by X^*, is defined as follows:

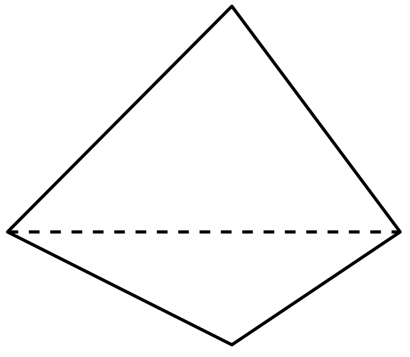
$$X^* = \{y \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } x \in X\}.$$

Illustration for the dual set X^*

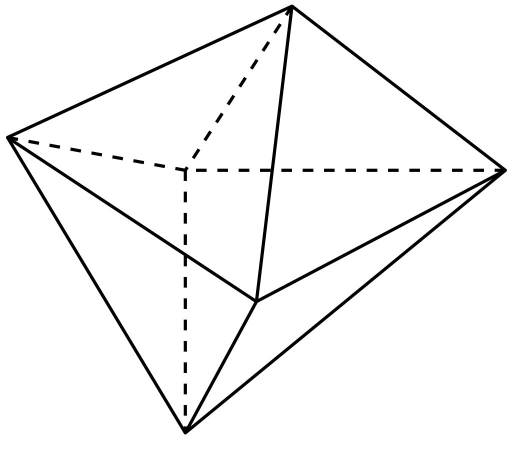
- Geometrically, X^* is the intersection of all half-spaces of the form $D_0(x)^-$ with $x \in X$.
- In other words, X^* consists of the origin plus all points y such that $X \subseteq D_0(y)^-$.
- For example, if X is the quadrilateral $a_1a_2a_3a_4$ shown above, the X^* is the quadrilateral $v_1v_2v_3v_4$.
- X^* is convex and closed and contains the origin.
- $(X^*)^*$ is the convex hull of $X \cup \{0\}$



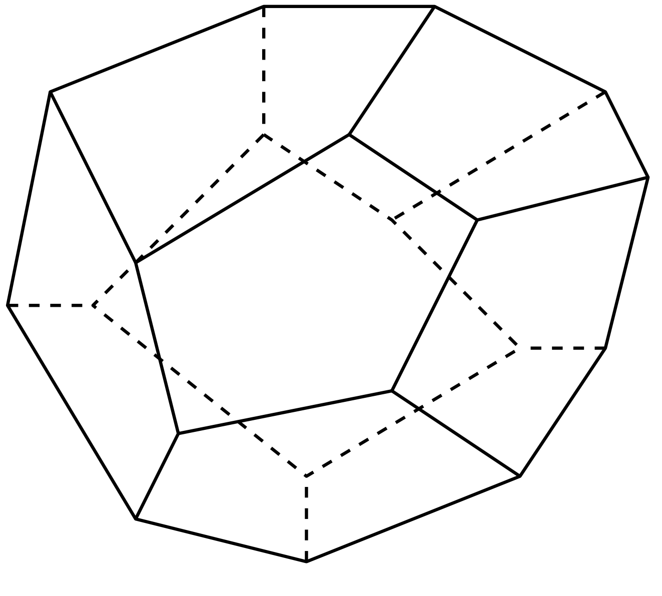
Famous convex polytopes in $\mathbb{R}^3$



Tetrahedron
four triangles
6 edges
4 vertices



Octahedron
8 triangles
12 edges
6 vertices



Dodecahedron
12 pentagon
30 edges
20 vertices

Two Types of Convex Polytopes

H-polyhedron/polytope

An H-polyhedron is an intersection of finitely many closed half-spaces in $\mathbb{R}^d$. An H-polytope if an bounded H-polyhedron.

V-polytope

An V-polytope is the convex hull of a finite point set in $\mathbb{R}^d$

Theorem

Each V-polytope is an H-polytope, and each H-polytope is a V-polytope.

Mathematically Equivalence, Computational Difference

- Whether a convex polytope is given as a convex hull of a finite point set or as an intersection of half-spaces
- Given a set of n points specifying a V-polytope, how to find its representationsa as an H-polytope?
- The number of required half-spaces may be astronomically larger than the number n of points

Another Illustration

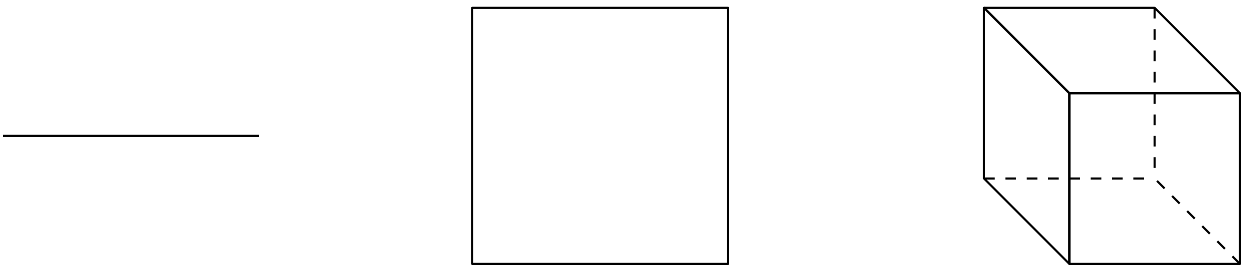
- Consider the maximization of a given linear function over a given polytope.
- For V-polytopes, it suffices to substitute all points of V into the given linear function and select the maximum of the resulting values
- For H-polytopes, it is equivalent to solving the problem of linear programming.

Dimension of a convex polyhedron P

- Dimension of its affine hull
- Smallest dimension of an Euclidean space containing a congruent copy of P

Cubes

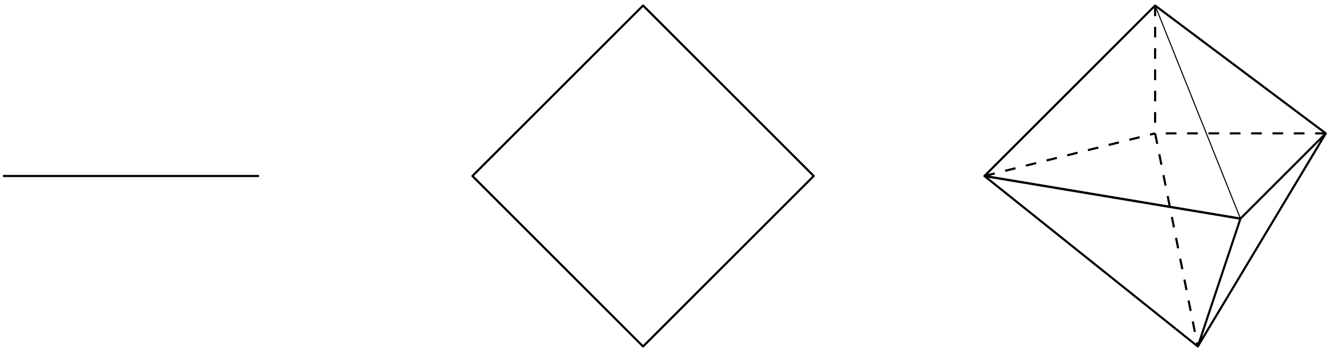
- The d-dimensional cube as a point set of the Cartesian Product $[-1, 1]^d$
- As a V-polytope, the d-dimentional cube is the convex hull of the set $\{-1, 1\}^d$ (2^d points).
- As a H-polytope, it is described by the inequalities $-1 \leq x_i \leq 1$, $i = 1, 2, \ldots, d$, i.e., by the intersection of $2d$ half-spaces
- 2^d points vs. $2d$ half-spaces
- The unit-ball of the maximium norm $||x||_\infty = \max_i |x_i|$



$d = 1$ $d = 2$ $d = 3$

Crosspolytope

- V-polytope: Convex hull of the "coordinates cross," i.e., the convex hulll of e_1, $-e_1$, e_2, $-e_2$, ..., e_d, and $-e_d$, where $e_1, \ldots, e_d$ are vectors of the stanard orthonormal basis. For $d = 2$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$.
- H-polytope: Intersection of 2^d half-spaces of the form $\langle \sigma, \leq \rangle 1$, where σ ranges over all vectors in $\{-1, 1\}^d$.
- $2d$ points vs. 2^d half-spaces
- Unit ball of l_1-norm $||x||_1 = \sum_{i=1}^{d} |x_i|$.

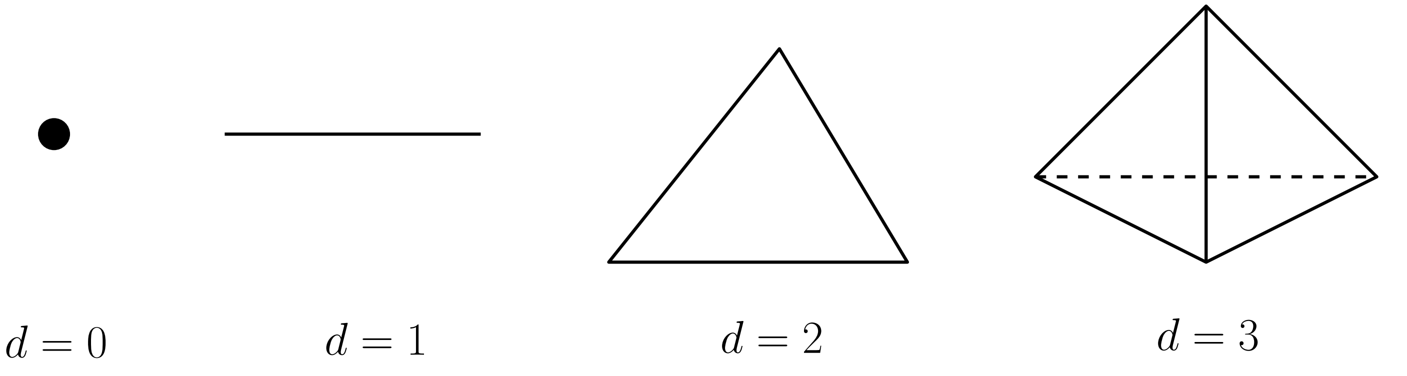


$d = 1$ $d = 2$ $d = 3$

Simplex

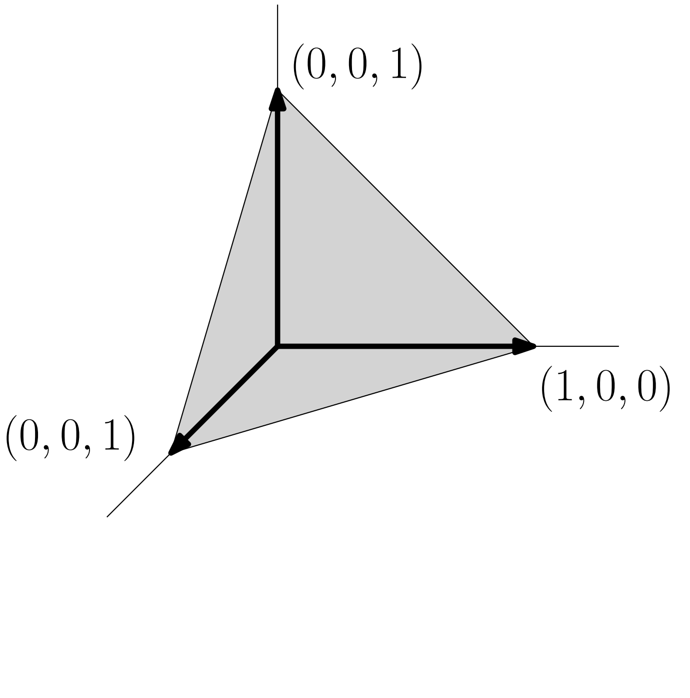
A *simplex* is the convex hull of an affinely independent point set in some $\mathbb{R}^d$

- A d-dimensional simplex in $\mathbb{R}^d$ can also be an intersection of $d+1$ half-spaces.
- The polytopes with smallest possible number of vertices (for a given dimension) are simplices.



A *regular* d-dimensional simplex in $\mathbb{R}^d$ is the convex hull of $d + 1$ points with all pairs of points having equal distances.

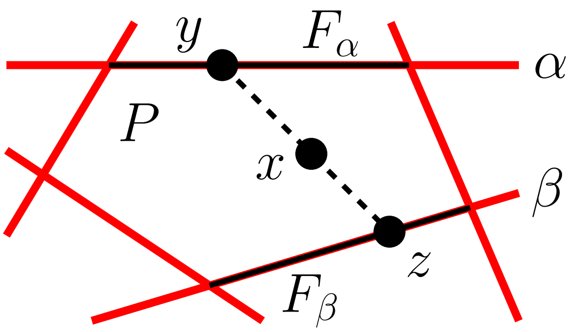
- Do not have a very nice representation in $\mathbb{R}^d$
- Simplest representation lives one dimension higher
- The convex hull of the $d+1$ vectors $e_1, \ldots, e_{d+1}$ of the standard orthonormal basis in $\mathbb{R}^{d+1}$ is a d-dimensional regular simplex with side length $\sqrt{2}$.



Proof of equivalence of H-polytope and V-polytope

=> (Let P be an H-polytope)

- Assume $d \geq 2$ and let Γ be a finite collection of closed half-spaces in $\mathbb{R}^d$ such that $P = \bigcap \Gamma$ is nonempty and bounded (By the induction, $(d-1)$ is correct)
- For each $\gamma \in \Gamma$, let $F_\gamma = P \cap \partial\gamma$ be the intersection of P with bounding hyperplane of γ.
- Each nonempty F_γ is an H-polytope of dimension of at most $(d-1)$, and it is the convex hull of a finite set $\boldsymbol{V_\gamma} \subset F_\gamma$ (by the inductive hypothesis)
- Claim $P = \text{conv}(V)$, where $V = \bigcup_{\gamma\in\Gamma} V_\gamma$
  - Let $x \in P$ and let l be a line passing through x.
  - The intersection $l \cap P$ is a segement, so let y and z be its endpoints
  - There are $\alpha, \beta \in \Gamma$ such that $y \in F_\alpha$ and $z \in F_\beta$
  - We have $y \in \text{conv}(V_\alpha)$ and $z \in \text{conv}(V_\beta)$.
  - $x \in \text{conv}(V_\alpha \bigcup V_\beta) \subseteq \text{conv}(V)$



(<=) (Let P be a V-polytope)

- Let $P = \text{conv}(V)$ with V finite and assume 0 is an interior point of P
- Consider the dual body $P^* = \bigcap_{v\in V} D_0(v)^-$
- Since P^* is an H-polytope, P^* is a V-polytope (what we just prove)
  - P^* is the convex hull of a finite point set U
- Since $P = (P^*)^*$, P is the intersection of finitely many half-spaces
  - $P = \bigcap_{u\in U} D_0(u)^-$