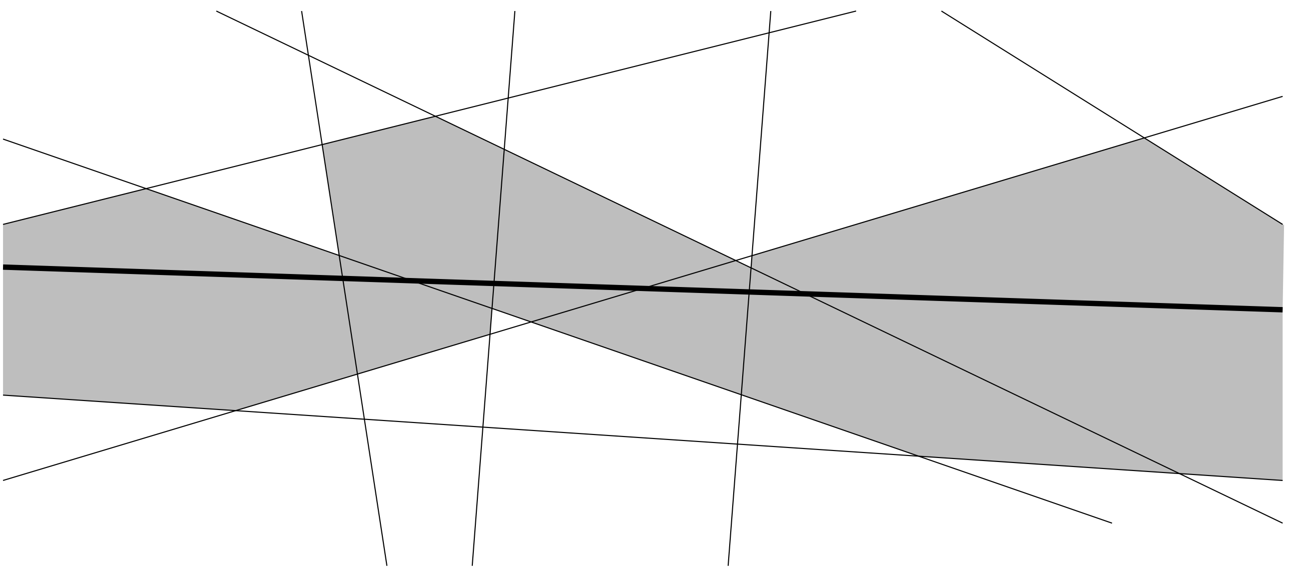
# **The Zone Theorem** (Chapter 6.4)

## Zone

Let $H$ be a set of $n$ hyperplane, and let $g$ be a hyperplane that may or may not lie in $H$.

The **zone** of $g$ is the set of the faces in the arrangement of $H$ that can see $g$

- Imagine the hyperplanes of $H$ are *opaque*
- A face $F$ can see the hyperplane $g$ if there are points $x \in F$ and $y \in g$ such that the open segment $\overline{xy}$ is not intersected by any hyperplane of $H$
  - It does not matter which point $x \in F$ we choose.
  - Either all of them can see $g$ or non can.



## Zone Theorem

The number of faces in the zone of any hyperplane in an arrangement of $n$ hyperplanes in $\mathbb{R}^d$ is $O(n^{d-1})$, with the constant of proportionality depending on $d$.

Before the proof

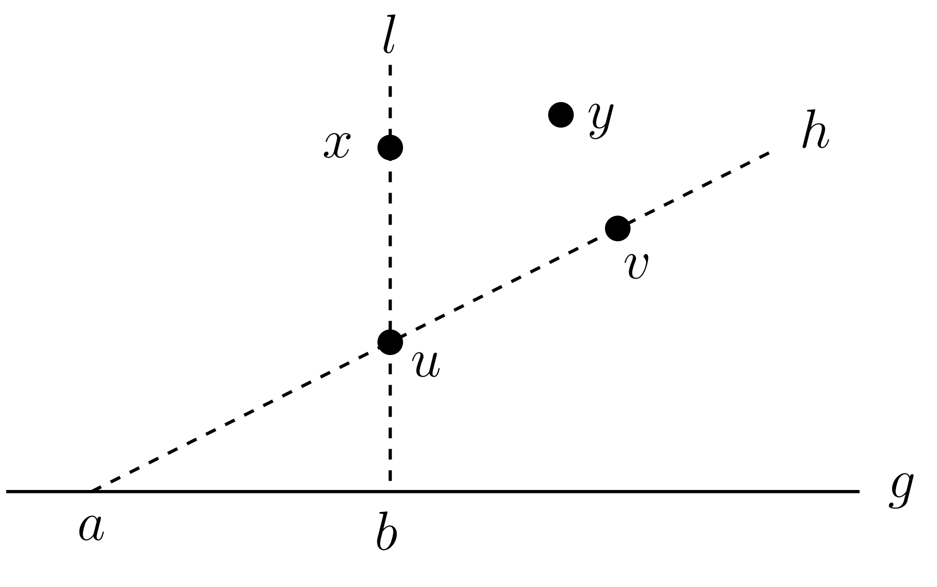
- Assume $H \cup \{g\}$ to be in general position
- The zone has $O(n^{d-1})$ cells because each $(d-1)$-dimensional cell of the $(d-1)$-dimensional arrangement within $g$ intersects only one $d$-dimensional cell of the zone
  - But it does not apply to the number of faces and vertices.

**Proof of Zone Theorem**

General Idea: Induction on the dimension $d$.

The case $d = 2$

- Let $H$ be a set of $n$ lines in the planes in general position
- Consider the zone of a line $g$
- Since a convex polygon has the same number of vertices and edges, it suffices to bound the total number of edges visible from the line $g$
- Imagine $g$ drawn horizontally and count the number of visible edges lying above $g$
  - At most $n$ visible edges intersect the line $g$ since each line of $H$ gives rise to at most one such edge
  - The other visible edges are disjoint from $g$
- Consider an edge $\overline{uv}$ from $g$ and visible from a point of $g$.
- Let $h \in H$ be the line containing $\overline{uv}$, and let $a$ be the intersection of $h$ with $g$
  - Assume $u$ is closer to $a$ than $v$ is.
- Let $l$ be the second line defining the vertex $u$, and let $b$ be the intersection of $l$ with $g$.
- Call $\overline{uv}$ a *right edge* of $l$ if $b$ lies to the right of $a$, and a *left edge* if $b$ lies to the left of $a$.

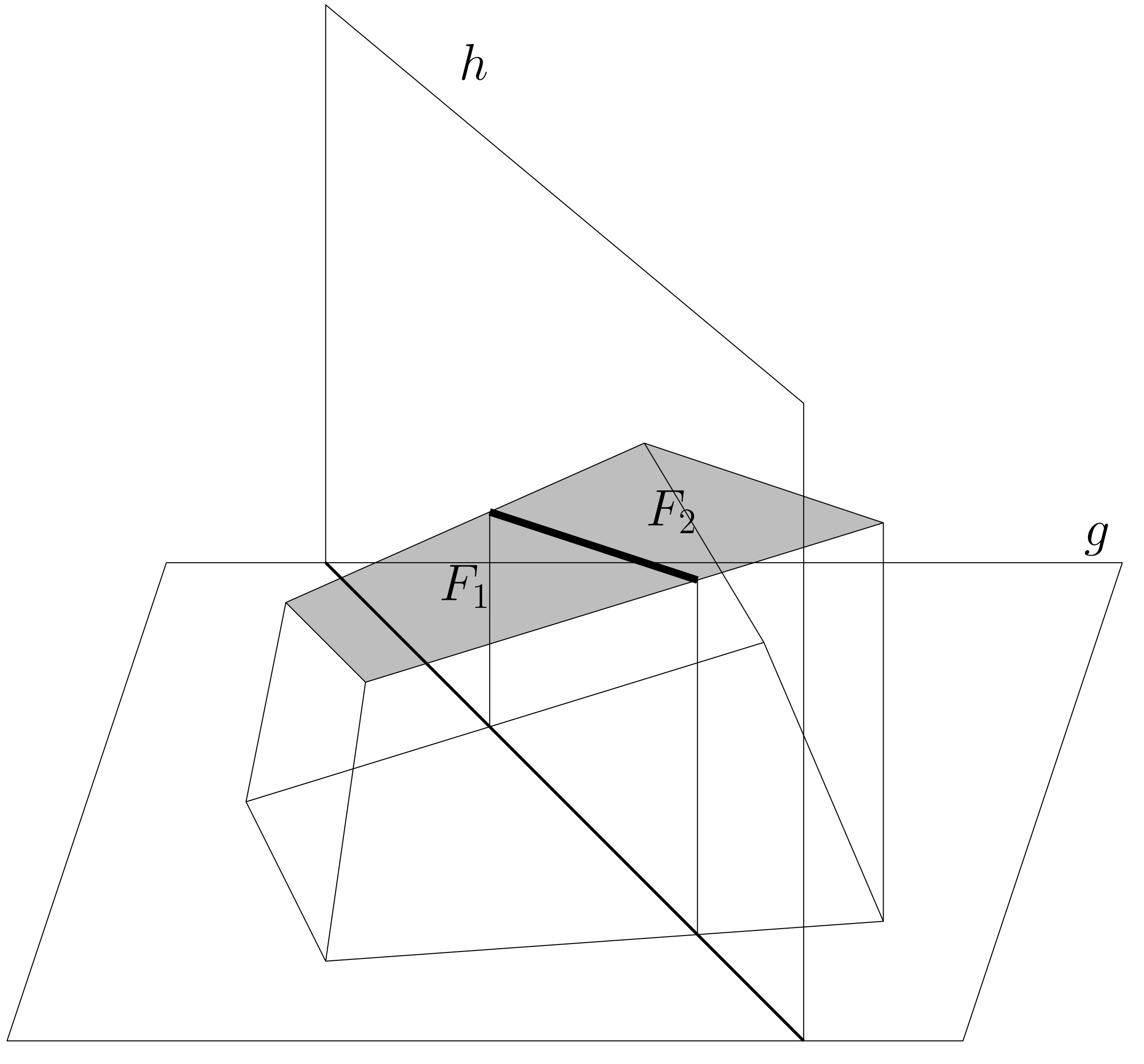


- Show that for each line $l$ there exists at mos one right edge
  - If it were not the case, there would exist two edges, $\overline{uv}$ and $\overline{xy}$, where $u$ lies lower than $x$, which would both be the right edges of $l$
  - The edge $\overline{xy}$ should see some point of the line $g$.
  - However, the part of $g$ lying to the right of $a$ is obscured by line $h$, and the part left of $a$ is obscured by the line
  - There exists a contradiction
- The number of visible edges is $O(n)$
  - the numer of right edges is at most $n$, so is the number of left edges
  - The visible edges lying below $g$ are counted symmetrically.

The case for $d > 2$

- Make the inductive step from $d - 1$ to $d$
  - Assume the total number of faces of a zone in $\mathbb{R}^{d-1}$ to be $O(n^{d-2})$, and want to bound the total number of zone faces in $\mathbb{R}^d$
- Use an averaging argument
  - begin with the slightly simpler case of counting only the facets (i.e., (d-1)-faces) of the zone
- Let $f(n)$ denote the maximum possible number of $(d-1)$-faces in the zone of an arrangement of $n$ hyperplanes in $\mathbb{R}^d$.
- Let $H$ be an arrangement and $g$ be a base hyperplane such that $f(n)$ is attained for them.
- Consider the folowing random experiment
  - Color a randomly chosen hyperplane $h \in H$ *red* and the other hyperplanes of H *blue*
  - Investigate the expected number of blue facets of the zone, where a facet is blue if it lies in a blue hyperplane.
- On one hand, since any facet has probability $\frac{n-1}{n}$ of becoming blue, the expcted number of blue facets is $\frac{n-1}{n}f(n)$.

- Bound the expected number of blue facets in different way
  - Consider the arrangement of blue hyperplanes
  - Its has at most $f(n-1)$ blue facets in the zone by the inductive hypothesis
  - Add the red plane, and look by how much the number of blue facets in the zone can increase
- A new blue facets can arise by adding the red hyperplane only if the red hyperplane slice some existing blue facets $F$ into two parts $F_1$ and $F_2$



- This operation increases the number of blue facets in the zone only if both $F_1$ and $F_2$ are visible from $g$.
  - In such a case, we look at the situation within the hyperplane $h$
- Claim $F \cap h$ are visible from $g \cap h$
  - Let $C$ be a cell of the zone in the arrangement of the blue hyperplanes having $F$ on the boundary
  - Exhibit a segment connecting $F \cap h$ to $g \cap h$ within $C$.
    * If $x_1 \in F_1$ see a point $y_1 \in g$ and $x_2 \in F_2$ see a point $y_2 \in g$, then the whole interior of the tetrahedron $x_1x_2y_1y_2$ is contained in $C$
    * The intersection of this tetrahedron with the hyperplane $h$ contains a segment witnessing the visibility of $g \cap h$ from $F \cap h$.
- We can obtain the following inequaility:
$$\frac{n-1}{n} f(n) \leq f(n-1) + O(n^{d-2}).$$
  - If we intersect all the blue hyperplanes and the hyperplane $g$ with the red hyperplane $h$, we get a $(d-1)$-dimensional arrangement, in which $F \cap h$ is a facet of the zone of the $(d-2)$-dimensional hyperplane $g \cap h$.
  - By the inductive hypothesis, this zone has $O(n^{d-2})$ facets.
  - Hence adding $h$ increases the number of blue facets of the zone by $O(n^{d-2})$.
- The previous considerations can be generalized to $(d-k)$-faces, where $1 \leq k \leq d-2$:
$$\frac{n-k}{n} f_{d-k}(n) \leq f_{d-k}(n-1) + O(n^{d-2}),$$
  - where $f_j(n)$ denote the maximum possible number of $j$-faces in the zone for $n$ hyperplanes in $\mathbb{R}^d$.
- Use the subsitituion $\phi(n) = \frac{f_{d-k}(n)}{n(n-1)\cdots(n-k-1)}$, and transform the recurrence into
$$\phi(n) \leq \phi(n-1) + O(n^{d-k-2}).$$
- For $k < d-1$, $\phi(n) = O(n^{d-k-1})$, and hence $f_{d-k}(n) = O(n^{d-1})$

- Still need to bound the case $k = d - 1$
- The number of vertices of the zone is at most proportional to the number of the 2-faces of the zone
  - Every vertex is contained in some 3-face of the zone
  - Within each such 3-face, the number of vertices is at most 3 times the numer of 2-faces, because the 3-faces is a 3 dimensional convex polyhedron.
  - Since our arrangement is simple, each 2-face is contained in a bouneded number of 3-faces
  - Therefore, the total number of vertices is at most proportional to $f_2(n) = O(n^{d-1})$
  - The analogous bound for edges follows immediately from the bound for vertices