Rheinische Friedrich-Wilhelms-Universität Bonn
Mathematisch-Naturwissenschaftliche Fakultät

Theoretical Aspects of Intruder Search

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Elmar Langetepe

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The manuscript will be successively extended during the lecture in the Wintersemester. Hints and comments for improvements can be given to Elmar Langetepe by E-Mail `elmar.langetepe@informatik.uni-bonn.de`. Thanks in advance!

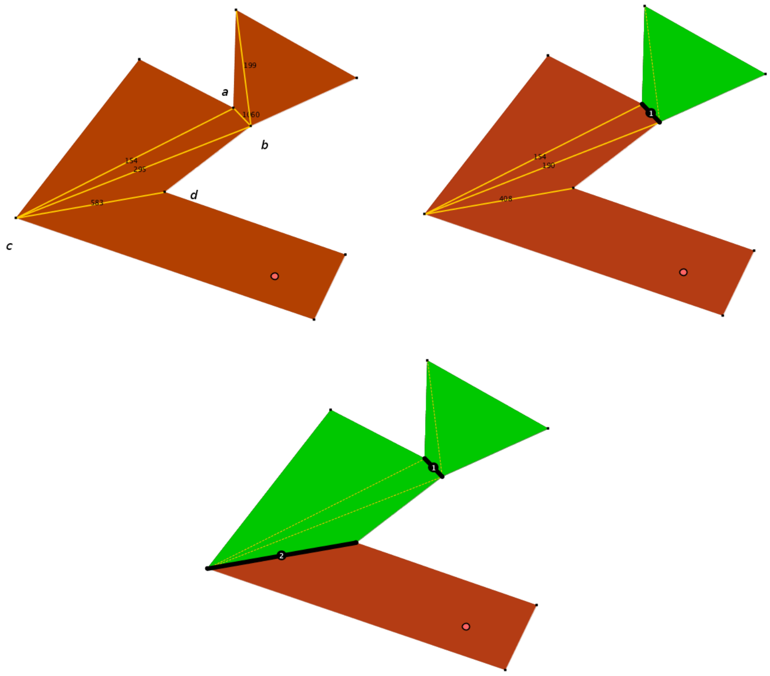

Figure 5.3: After the usage of barrier (a, b), the priority of barrier (c, d) decreases because (a, b) already protects parts of the area related to (c, d).

by a single barrier. In this case we can protect new areas A' which are not related to single barriers. This is obviously not represented in the algorithm and the analysis.

In principle the algorithm could also run for intersecting barriers but there is one major problem. The execution time for building a barrier is not independent from the order of the barrier constructed so far. A barrier might have a release times that depends on the barriers constructed before. For example a barrier b_1 blocks the fire and extends the release time of a barrier b_2 that intersects b_1.

So there might be a barrier of the optimal solution, that contributes to the optimal profit with an arbitrary large amount of its profit but cannot be scheduled in the approximation since it depends on the construction of a very less important (w.r.t. area profit) barrier. The analysis might fail.

Exercise 22 *Construct an example, where the construction of a barrier b_2, depends on the construction of a barrier b_1.*

Exercise 23 *If intersections are allowed, where is the main problem in the proof of the above approximation result?*

Finally, we end the section with an example where the profit of a job is reduced by overlapping, as shown in Figure 5.3.

5.3 Firefighting in the plane

In this section we consider the case that the fire spreads in the plane and we would like to build a firebreak. We assume that there is a single fire that spreads with the same unit speed

speed into any direction – an expanding circle. The configuration was already considered in the Introduction 1.1.3 for circle shaped firebreaks.

Let us assume that we can build arbitrary barrier curves in the plane with speed $v > 1$. The current point has to keep outside the current fire extension. Already constructed parts of the curve serve as obstacles for the expanding fire.

Geometric Firefighter Problem in the plane
Instance: An expanding fire-circle that spreads with unit speed from a given starting point s in the plane and has already reached radius A.
Ouestion: How fast must a firefighter be, to build a firebreak that finally fully encloses and stops the expanding fire?

We assume that we build a single connected curve with a single firefighter.

5.4 On finding an upper bound for speed v

First we discuss some structural properties. If we start a the boundary of the fire and move with speed v, the curve that keeps as close as possible to the fire is a logarithmic spiral with excentricity α for $v = \frac{1}{\cos\alpha}$.

This can be easily seen by the following argument. For a fireman close to the fire, locally not all movements are safe. If the fireman performs an infinitesimal small step x at some point p very close to the fire, the fire grows by distance $\frac{x}{v} = x\cos\alpha$ for $v = \frac{1}{\cos\alpha}$ from the source; see Figure 5.4. This means that with respect to the ray R_q emanating from the source B and running through q the fire fighters can leave R_q *locally* from q only in a direction smaller than or equal to α. If the firefigther continuously exactly move into the direction α, he will follow a logarithmic spiral with excentricity α around B.

In polar-coordinates a logarithmic spiral S is defined by $S(\varphi) := (\varphi, a \cdot e^{\varphi\cot\alpha})$ where a is a constant defined by the distance of the spiral to its origin for the angle $\varphi = 0$. The spiral has the property that the tangent to the spiral at some point $p \in S$ and the ray emanating from the origin and running through p build an angle α for all points $p \in S$. Furthermore, for an origin B and two points q and p on S the path, S_q^p, along S has length

$$|S_q^p| = \frac{1}{\cos\alpha}\left(|Bq| - |Bp|\right) . \tag{5.8}$$

This means that a logarithmic spiral of excentricity α that starts at some point q close to the fire circle of radius $|Bq|$ is always a correct firebreak. Additionally, the logarithmic spiral with excentricity $\alpha \in (0, \pi/2]$ that starts at some point q will always finally hit a line l' running through q; see Figure 5.4.

Exercise 24 *Let us assume that a natural firebreak is already given by a line and the fire expands in one half-plane. How fast must the firefighter be, in order to enclose the fire in this scenario?*

5.4.1 The limit speed

The following considerations are take from Bresson et al. 2008. The above discussion means that logarithmic spirals might be good candidates for enclosing an expanding circle.

We consider the following situation for a given spiral excentricity $\beta \in (\pi/2, 0)$ (for which $\lambda_\beta := \cot\beta$ runs from 0 to infinity). Let us assume that a spiral firebreak was constructed from the origin up to point D with some speed v and reaches D at some time t. Now the fire starts and also mets the point D after the same time t.

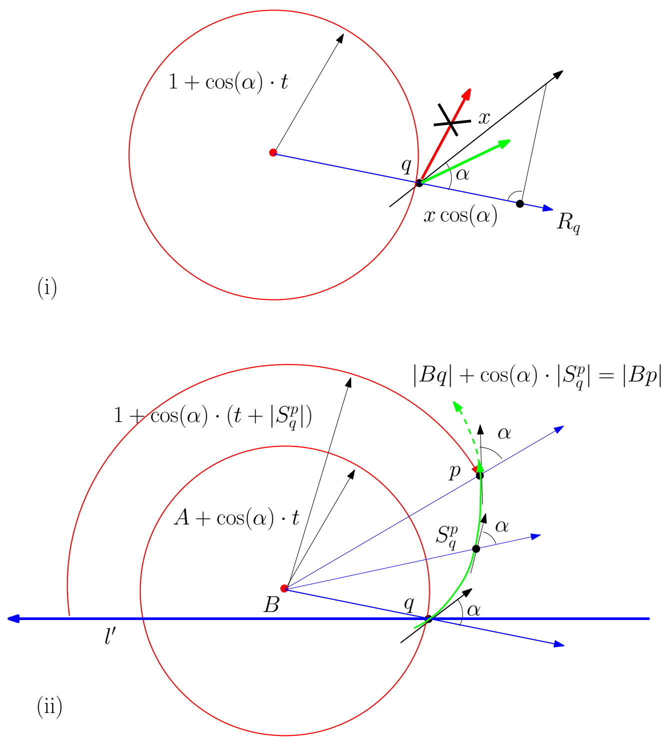

Figure 5.4: (i) At some point q close to the fire, locally the firefigther can leave the ray R_q at q only in a direction smaller than or equal to α. (ii) If the firefigthers continuously exactly move into direction α, they follow a logarithmic spiral with excentricity α around B which results in a correct firebreak.

Intuitively, we assume that the fire starts with some small delay and runs along the already constructed spiral to the point $p = (0, 1)$ and then follows the tangent at p and always reaches the already constructed spiral arc of the next round at some point $D = (2\pi + \gamma, a \cdot e^{(2\pi+\gamma)\cot\beta})$ as depicted in Figure 5.5. For the delay in the very beginning we obviously have to start the fire only one round later, below we would like to construct a correct starting situation. Since we can scale the situation arbitrarily, the overall absolute delay can be arbitrarily small and will be covered by speed. We would like to find a necessary condition for the speed for moving on.

By the law of sine we calculate the corresponding angle $\gamma := \gamma(\beta) \in (0, \pi/2)$ implicity by

$$\frac{a \cdot e^{(2\pi+\gamma)\cot\beta}}{\sin\beta} = \frac{a}{\sin(\beta - \gamma)} \iff e^{(2\pi+\gamma)\cot\beta} = \frac{\sin\beta}{\sin(\beta-\gamma)}\,. \tag{5.9}$$

By construction the angle γ exists for any spiral and it grows continuously in the interval $(0, \pi/2)$ for β in $(0, \pi/2)$. We would like to compute, how fast we have to move along the spiral so that the fire following us along the spiral will always exactly met us at point D. (Speed less than this value will let the fire overun us at point D.)

This means, that we consider the ratio

$$f(\beta) := \frac{|\text{Length of the spiral from } O \text{ to } D|}{|\text{Length of the spiral from } O \text{ to } C| + |\text{Length of the segment } CD|}\,. \tag{5.10}$$

We express CD by $a\frac{\sin\gamma}{\sin\beta}e^{(2\pi+\gamma)\cot\beta}$ by the law of sine and Equation 5.10 reads as follows:

$$f(\beta) = \frac{\frac{1}{\cos\beta}e^{(2\pi+\gamma)\cot\beta}}{\frac{1}{\cos\beta} + \frac{\sin\gamma}{\sin\beta}e^{(2\pi+\gamma)\cot\beta}} = \frac{\frac{1}{\cos\beta}\frac{\sin\beta}{\sin(\beta-\gamma)}}{\frac{1}{\cos\beta} + \frac{\sin\gamma}{\sin\beta}\frac{\sin\beta}{\sin(\beta-\gamma)}} = \frac{1}{\cos\gamma}\,. \tag{5.11}$$

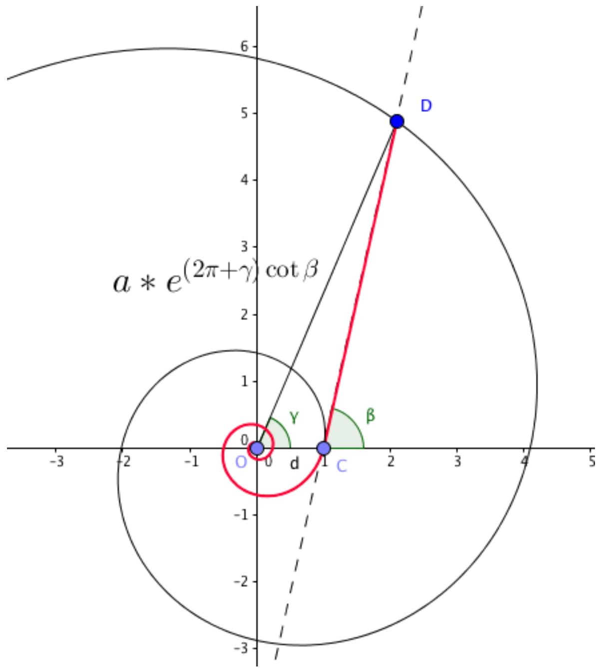

Figure 5.5: The fire starts with a delay of one round and follows the boundary of the spiral. How fast do we have to move so that the fire cannot overrun us at point D? Let $a = \overline{OC}$. By the definition of the spiral and the law of sine we can compute the corresponding lenghts.

Let us consider the behaviour of $f(\beta)$ at the boundary. If β goes to 0 and $\cot\beta$ goes to infinity, we argue that $\gamma(\beta)$ also goes to zero, which means

$$\lim_{\beta\mapsto 0} f(\beta) = \lim_{\gamma\mapsto 0} \frac{1}{\cos\gamma} = 1\,. \tag{5.12}$$

If β goes to $\pi/2$ from below, $\cot\beta$ goes to 0 but remains positive, $\gamma(\beta)$ is smaller than $\pi/2$. Thus,

$$\limsup_{\beta\mapsto\pi/2\nearrow} f(\beta) \le \limsup_{\beta\mapsto\pi/2\nearrow} e^{(2\pi+\gamma)\cot\beta} \le \lim_{\beta\mapsto\pi/2\nearrow} e^{(5\pi/2)\cot\beta} = 1\,. \tag{5.13}$$

This means that f is a continuous function in $\beta = (0, \pi/2)$ and there will be some unique global maximum

$$v_l := \max_{\beta\in(0,\pi/2)} f(\beta)\,.$$

This global maximum is found numerically by $\beta_l = 1.29783410242\ldots$ and gives $v_l = f(\beta_l) = 2.614430844\ldots$ and $\gamma(\beta_l) = 1.178303978\ldots$ A sketch of the plot of $f(\beta)$ is shown in Figure 5.6.

This means that in the above scenario, we can assume that if we construct a spiral of some arbitrary excentricity β with speed $v > v_l$ for t time steps and after that let the fire start at the origin and spread also time t, the fire will keep really behind the current point F as depicted in Figure 5.7.

Now we would like to argue that for speed $v > v_l$ we always also find a correct starting situation that resembles the firefighting problem.

1. The starting situation

Let us assume that we have speed $v_1 = \frac{1}{\cos\beta_1} > v_l$ and we use a spiral with excentricity β_1. Then there is always a speed $v' = \frac{1}{\cos\beta'}$ with $v_1 > v' > v_l$ which allows us to use the spiral for

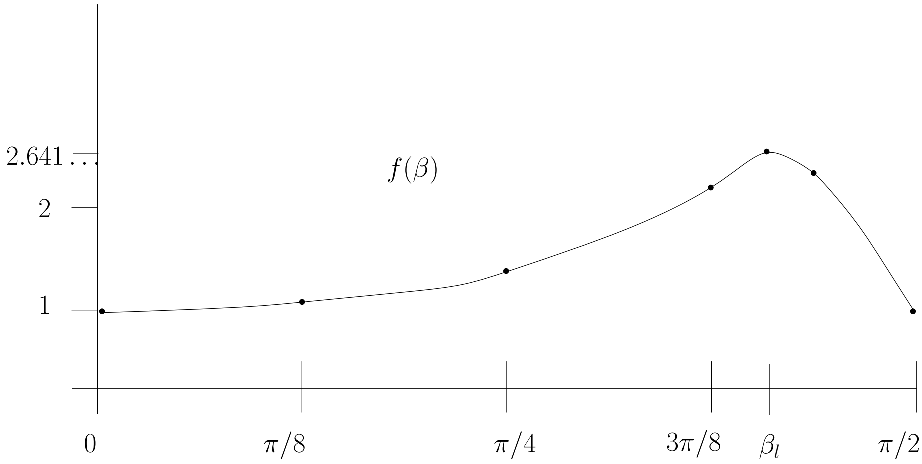

Figure 5.6: A sketch of the plot of the curve $f(\beta)$ for $\beta \in (0, \pi/2)$. There is a unique maximum 2.614430844 ... for $\beta_l = 1.29783410242\ldots$

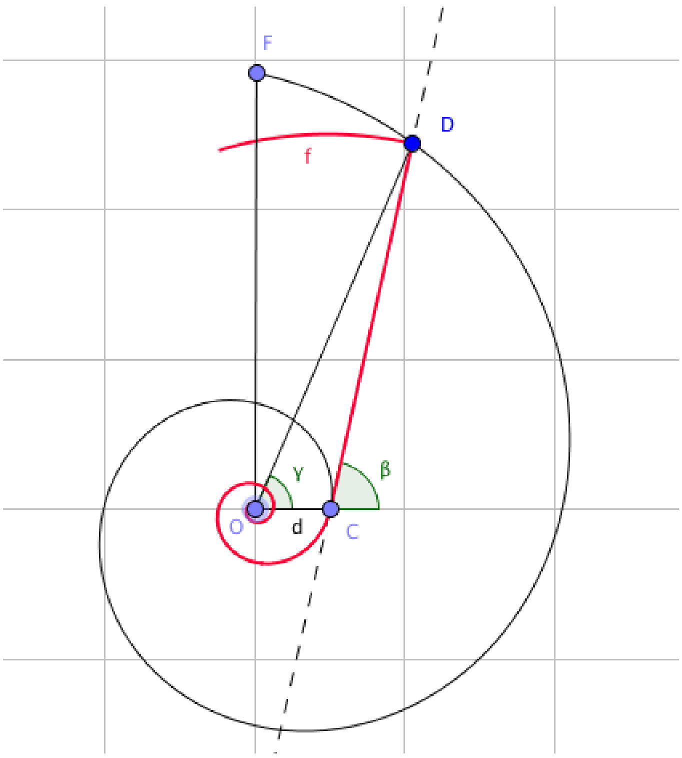

Figure 5.7: For speed $v > v_l$ the fire keeps behind the current point F.

β_1 as follows. If the firefighter moves with speed $v' < v_1$ for the common start from the origin the fire will not reach the current point F

Consider Figure **??**. Let us assume that for some time t' and speed $f(\beta_1)$ both, firefighter and fire met point F by starting at the origin. Thus, for speed $v' = \frac{1}{\cos\beta'}$ and meeting F the fire would have reached some point point D and with speed $\frac{1}{\cos\beta_1}$ some point N and the time difference x reads $t(\frac{1}{\cos\beta_1} - \frac{1}{\cos\beta'})$. More precisely, F is met at time $t_1 = t\cos\beta_1$ and time $t_2 = t\cos\beta_2$ and $t_1 - t_2$ gives the time difference for the fire, the time used for moving from N to D.

We use this time difference to construct the starting situation, so that the fire and the firefighter do not start at the same point. Thus we attain a correct starting situation that resembles the firefighting problem.

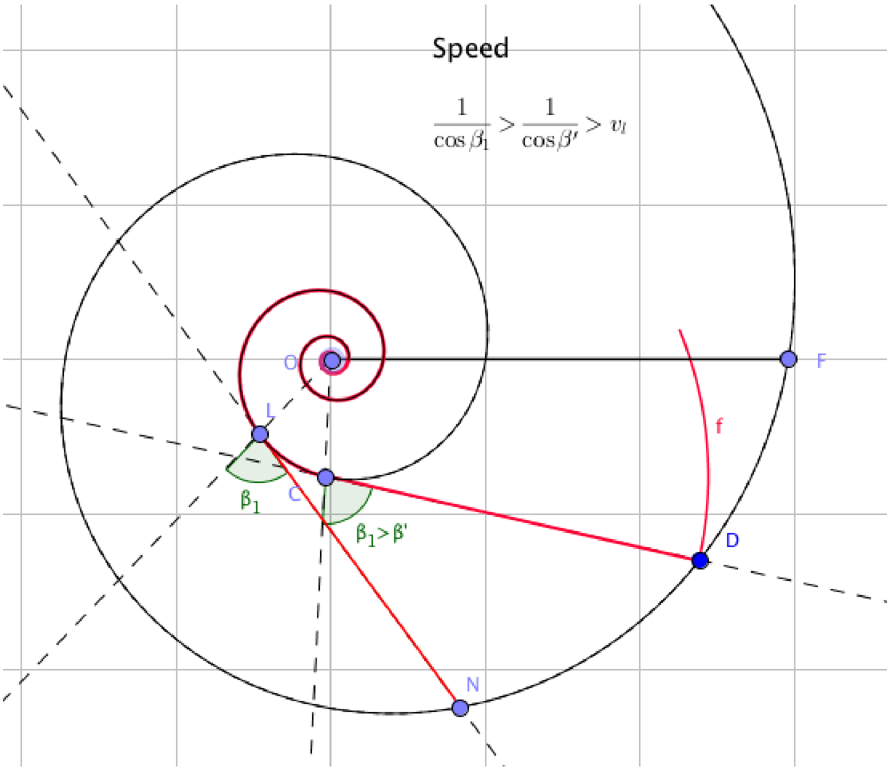

Figure 5.8: We can start the spiral with a slightly smaller speed. The current point is still not met and a distance x can be used for the starting situation.

There are always small values ϵ_1 and ϵ_2 so that

$$\frac{1}{\cos\beta'}(\epsilon_1 + \epsilon_2) - \epsilon_2 < x$$

holds. We allow the fire to start a bit earlier while we start away from the fire with speed $\frac{1}{\cos\beta_1}$. The fire should only reach D when the firefighter is located at F.

The starting situation is given in Figure 5.9. The fire has starting radius ϵ_1. We start with the spiral of excentricity β_1 at distance $\epsilon_1 + \epsilon_2$ away from the fire at point some point B. The firefighter starts at point B and the fire starts spreading at the same time. The fire will reach B after ϵ_2 time steps, and the firefighter have already moved a distance of $\frac{\epsilon_2}{\cos\beta_1}$. Additionally, the excentricity β_1 takes care, that we do not move into the fire by accident.

For speed $\frac{1}{\cos\beta'}$, the time saved by the fire is $\frac{1}{\cos\beta'}(\epsilon_1 + \epsilon_2) - \epsilon_2$. But this helps the fire only to visit point D, if we use speed $\frac{1}{\cos\beta_1}$.

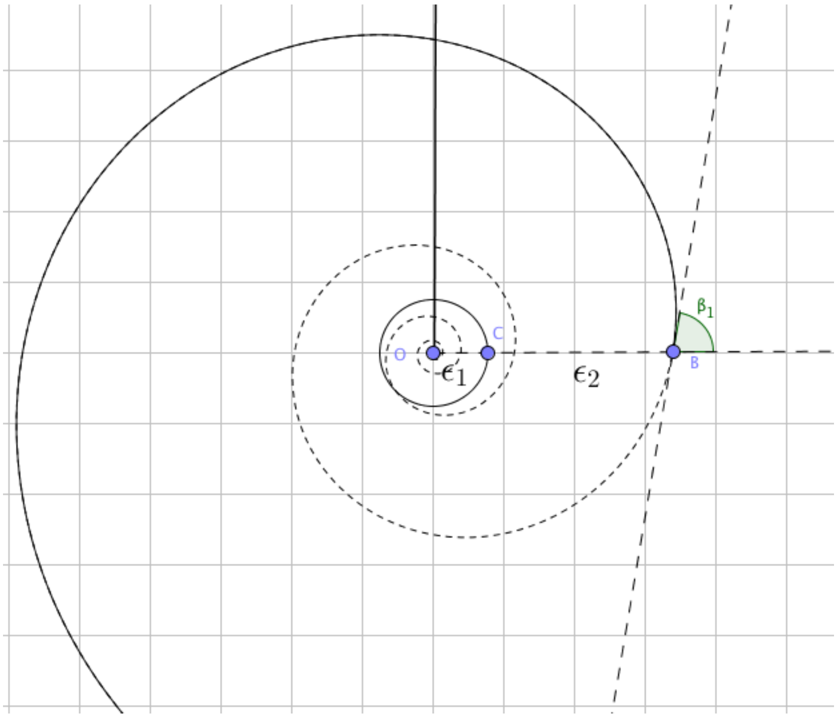

Figure 5.9: We make use of the time difference x and attain an admissable starting situation as given in Figure 5.10.

Altogether, we can assume that for speed $v_1 = \frac{1}{\cos\beta_1}$ we can start a spiral strategy outside the fire circle, and we reach some point $(e^{2k_1\pi\cot\beta_1}, 0)$ whereas the fire spreads up to some point D behind the current location F of the fire fighter; see Figure 5.10.

2. An iteration process for a successful strategy

Here we sketch the proof given by Bresson et al. 2008. The first spiral starts with radius $C_1 := \epsilon_1 + \epsilon_2$ and excentricity β_1 and ends at some point $F = (C_1 e^{2k_1\pi\cot\beta_1}, 0)$. The fire keeps behind at some point D as shown in Figure 5.11.

We now start another spiral at $F = (e^{2k_1\pi\cot\beta_1}, 0)$ with excentricity $\beta_2 > \beta_1$ and starting radius $C_2 = C_1 e^{2k_1\pi\cot\beta_1}$. This means that we make use of a slightly tigther logarithmic spiral. If β_2 is small enough, this spiral strategy will also be a correct intermediate strategy.

We let the spiral with excentricity β_2 run up to some angle $2k_2\pi$, thus attaining the same situation again. The strategy ends at some point F_2 and the fire is behind at some point D_2; see Figure 5.12.

After scaling, we assume that we have the same situation again. This means that we apply the same arguments and attain a sequence of increasing angles $\beta_1 < \beta_2 < \ldots < \beta_l$ and successive spirals $C_i e^{2\pi k_i \cot\beta_i}$ that result in a correct strategy.

Finally, we can also scale so that the overall strategy ends at point $(e^{2\pi\cot\beta_m}, 0)$. We would like to argue that for some β_m large enough there is time enough to move from $(e^{2\pi\cot\beta_m}, 0)$ to the point $(1,0)$ on the previous round. This means that the fire did not arrive at $(1,0)$; see Figure5.13.

The running time of the overall last spiral is smaller than $\frac{1}{\cos\beta_m} e^{2\pi\cot\beta_m}$. Additionally, moving to point $(1,0)$ yields $e^{2\pi\cot\beta_m} - 1$. For reaching point $(1,0)$ the fire also has to a distance

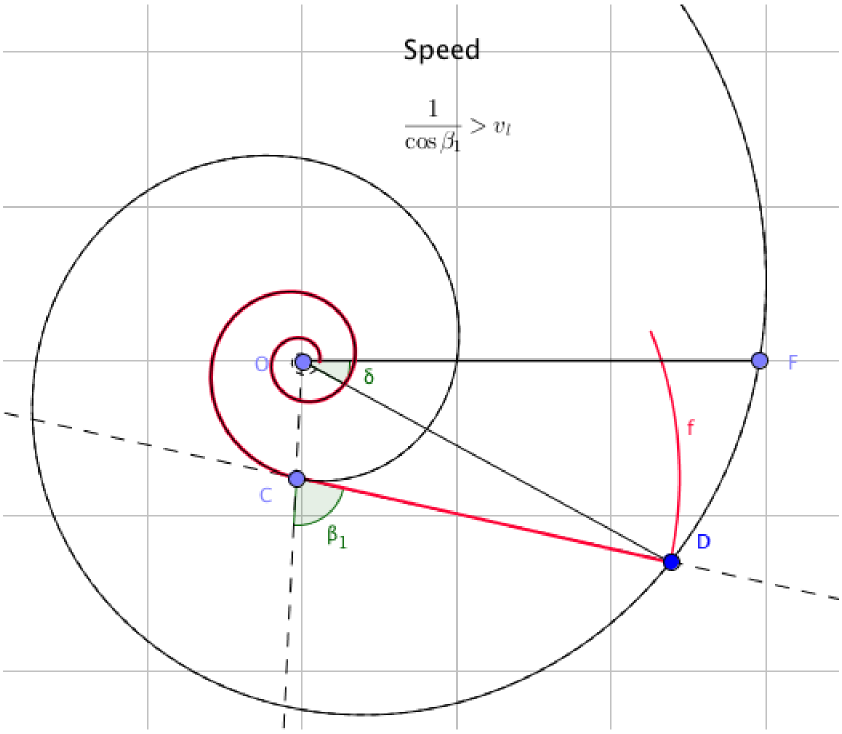

Figure 5.10: An admissable starting situation for speed $v_1 = \frac{1}{\cos \beta_1}$.

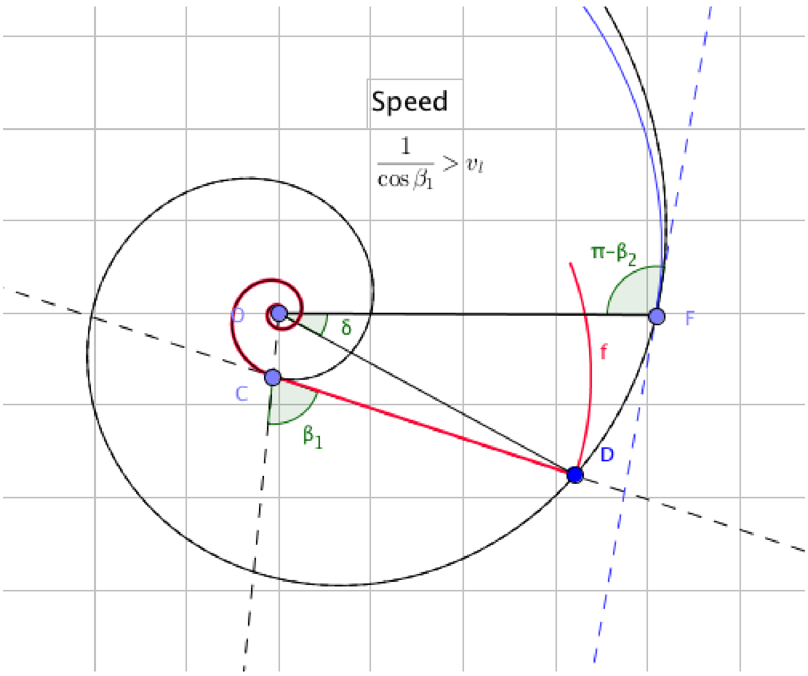

Figure 5.11: We start with a second spiral and excentricity $\beta_2 > \beta_1$.

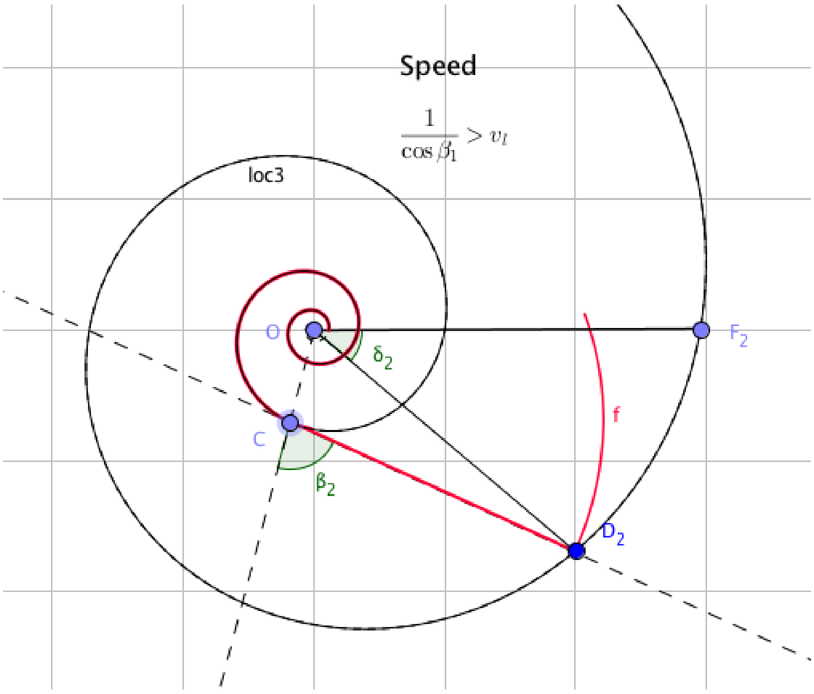

Figure 5.12: By scaling we attain the same situation again.

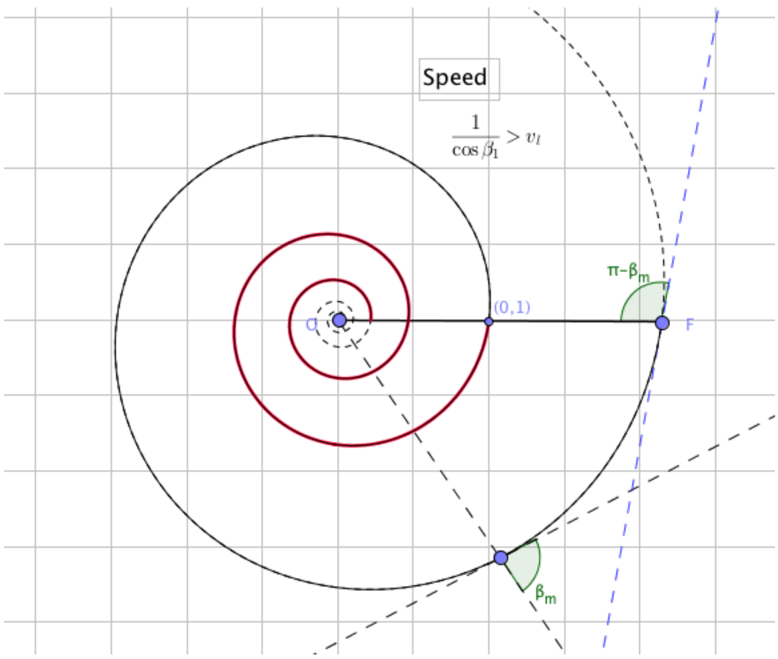

Figure 5.13: The firefighter ends with a spiral of excentricity $\beta_m > \beta_{m-1} > \cdots > \beta_1$. The angle β_m is large enough so that the fire on the previous round did not arrive at $(1, 0)$ when the firefighter moves from F to $(1, 0)$ along the line segment.

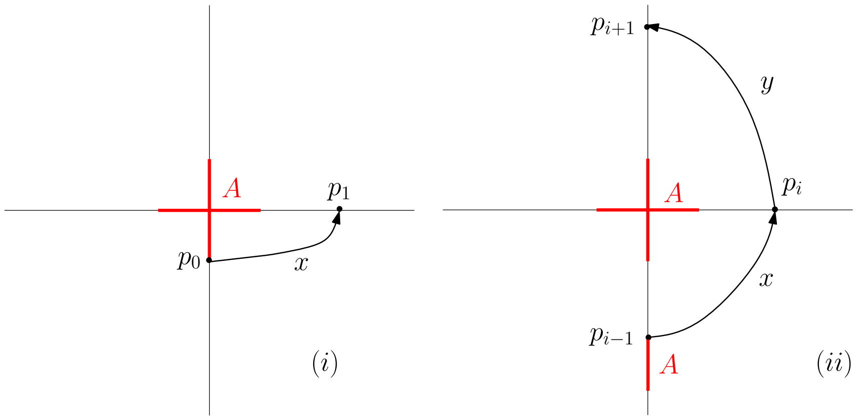

Figure 5.14: Proof of Theorem 57.

arbitrarily close to $\frac{1}{\cos\beta_m}$ for following the last spiral, this follows from scaling. If β_m grows, there will be some β_m that finally fulfills

$$\frac{1}{\cos\beta_m} > \frac{1}{\cos\beta_1}\left(\frac{1}{\cos\beta_m}e^{2\pi\cot\beta_m} + (e^{2\pi\cot\beta_m} - 1)\right) \tag{5.14}$$

which gives the conclusion. For example for $\beta_1 \approx 1.191388\ldots$ and $\frac{1}{\cos\beta_1} = 2.7$ we require $\beta_m > 1.4268$.

Theorem 56 *(Bresson et al. 2008) For any speed $v > v_l \approx 2.614430844$ there is a spiralling strategy that finally encloses an expanding circle that expands with unit speed.*

5.5 A simple lower bound for spiralling strategies

A barrier building strategy S is called *spiralling* if it starts on the boundary of a fire of radius A, and visits the four coordinate half-axes in counterclockwise order and at increasing distances from the origin.

Now let S be any spiralling strategy of maximum speed $v \leq (1+\sqrt{5})/2 \approx 1.618$, the golden ratio. We can assume that S proceeds at constant speed v. Let $p_0, p_1, p_2, \ldots$ denote the points on the coordinate axes visited, in this order, by S. The following Lemma shows that S cannot succeed because there is still fire burning outside the barrier on the axis previously visited.

Lemma 57 *Let A be the initial fire radius. When S visits point p_{i+1}, the interval $[p_i, p_i + \texttt{sign}(p_i)A]$ on the axis visited before is on fire.*

Proof. The proof is by induction on i. Suppose strategy S builds a barrier of length x between p_0 and p_1, as shown in Figure 5.14 (i). During this time the fire advances x/v along the positive X-axis, so that $A + x/v \leq p_1 \leq x$ must hold, or

$$\frac{x}{v} \geq \frac{1}{v-1}A > A;$$

the last inequality follows from $v < 2$. Thus, the fire has enough time to move a distance of A from p_0 downwards along the negative Y-axis.

Now let us assume that strategy S builds a barrier of length y between p_i and p_{i+1}, as shown in Figure 5.14 (ii). By induction, the interval of length A below p_{i-1} is on fire. Also, when the fighter moves on from p_i, there must be a burning interval of length at least $A + x/v$ on the positive Y-axis which is not bounded by a barrier from above. This is clear if p_{i+1} is the first point visited on the positive Y-axis, and it follows by induction, otherwise. Thus, we must have $A + x/v + y/v \leq p_{i+1} \leq y$, hence

$$\frac{y}{v} \geq \frac{1}{v-1}A + \frac{1}{v(v-1)}x \; > \; A + x.$$

The rightmost inequality follows since v is supposed to be smaller than the golden ratio, which satisfies $X^2 - X - 1 = 0$; hence, $v^2 - v < 1$. This shows that the fire has time to crawl along the barrier from p_{i-1} to p_i, and a distance A to the right, as the fighter moves to p_{i+1}, completing the proof of Theorem 58. □

Theorem 58 *In order to enclose the fire, a spiralling strategy must be of speed*

$$v \; > \; \frac{1+\sqrt{5}}{2} \approx 1.618,$$

the golden ratio.