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Mathematisch-Naturwissenschaftliche Fakultät

Theoretical Aspects of Intruder Search

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Elmar Langetepe

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The manuscript will be successively extended during the lecture in the Wintersemester. Hints and comments for improvements can be given to Elmar Langetepe by E-Mail `elmar.langetepe@informatik.uni-bonn.de`. Thanks in advance!

6.3 Continuous variant in simple polygons

In Section 6.1 we considered the optimal escape path for some special convex environment. For more complicated environments the escape path is still unknown and it is unrealistic to think that there will be much progress in the near future. Therefore we are searching for an escape path that can be computed easily and might serve as a substitute.

Assume that you are located within an unknown environment and would like to reach its boundary. Formally, for the environment we consider a closed Jordan curve B that subdivides the Euclidean plane into exactly two regions. The starting point s lies inside the inner region, say P. Starting from s the task is to find a point on the boundary B as soon as possible.

If you have some idea about the distance x from s to the boundary B but nothing more, it is very intuitive to move along the circle of radius x around the starting point. Therefore a reasonable strategy moves toward this circle along a shortest path (by radius x) in some direction and then follows the circle in either clockwise or counterclockwise direction until the boundary is met. Let us call this a *circular strategy*. If we hit the boundary after moving an arc of angle α_x along the cirlce the overall path length is given by $x(1 + \alpha_x)$.

We would like to use such a circular strategy of small path length. In the sense of a game, the adversary can only rotate the environment around the starting point and the certificate path guarantees to hit the boundary for any rotation.

6.3.1 Extreme cases and general definition

Let us first consider two somehow extreme examples of the above intuitive idea as given in Figure 6.10. If the distance from s to the boundary is almost the same in any direction (similar to a circle), a line segment with maximal distance to the boundary (roughly the radius of the circle) will always hit the boundary and is indeed a very good escape path for any direction; see Figure 6.10(ii). The movement along an arc is not necessary in this case. In other words α_x equals 0. We check a single direction for the largest distance.

On the other hand, if the distance to the boundary is very large w.r.t. almost all directions from s but is relatively small (distance x) for some few directions, a segment of length x and a circular arc of length $x\alpha_x$ with $\alpha_x \approx 2\pi$ will hit the boundary for any starting direction of the segment x; see Figure 6.10(i). The overall path length $x(1 + \alpha_x)$ is also relatively small. We check a small distance for many (almost all) directions. Figure 6.11 shows a sketch of the radial maximal distance function of the extreme examples in Figure 6.10.

Now consider a more general environment modeled by a simple polygon P and a fixed starting point s in P as given in Figure 6.12(ii). For convenience we first make use of an example where any boundary point b of P is *visible* from s, i.e. the segment sb lies fully inside P.

In general, for keeping the certificate simple, we always take the maximal distance to the boundary in any direction into account; see also Figure 6.13.

For any arbitrary polygon P and for any radial direction $\phi \in [0, 2\pi]$ from s we consider the boundary point $p_{s,\phi}$ on P with *maximal* distance from s in direction ϕ. This gives a *radial maximal distance function* $f(\phi) := |s\, p_{s,\phi}|$ as depicted in Figure 6.12(i).

Now let $p_{s,\phi}$ be a point with maximal distance $x := |s\, p_{s,\phi}|$ in direction ϕ. For any circle $C_s(x)$ with radius x around s such that $C_s(x)$ hits the boundary of P there will be some maximal arc $\alpha_s(x)$ so that the above simple circular strategy is successful independently from the starting direction for x. We are looking for the maximum circle segment of $C_s(x)$ that fully lies inside P.

Let $\Pi_s(x)$ denote this *certificate path for distance* x of maximal length $x(1 + \alpha_s(x))$. The interpretation is, that independently from the starting direction for x this finite path will always

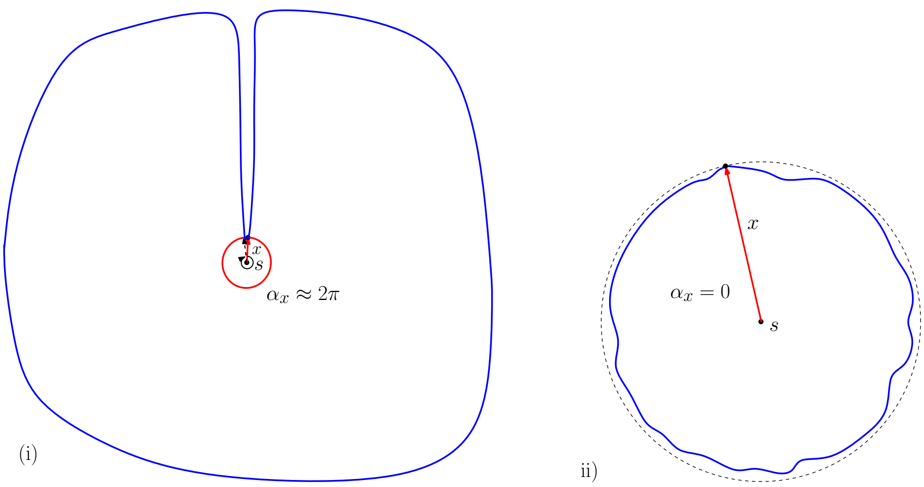

Figure 6.10: Two extreme situation for reaching the boundary with a circular arc. In the righthandside figure the radial maximal distance from s to the boundary is almost the same in any direction. So it suffices to move in an arbitrary direction of maximal distance which is optimal. In the lefthandside figure only the distance to some few boundary points is very small but much larger to most of the others. Therefore a reasonable path checks the small distance with a circular arc of length almost 2π. In both case $x(1+\alpha_x)$ is minimal among all such circular strategies. The distance functions of the two examples and the geometric interpretation of the reasonable escape paths is shown in Figure 6.11.

hit the boundary. The adversary can only rotate the environment for attaining a worst case length of $x(1+\alpha_x)$. The path $\Pi_s(x)$ can be translated into the radial maximal distance function setting. It consists of two segments, starting with a vertical segment of length x and ending with a horizonal segment of length α_x; see Figure 6.12(i). For any starting angle this path will hit the boundary of the distance function.

In turn, the overall *certificate path* Π_s in P for a given starting point s is the shortest certificate path $\Pi_s(x)$ among all distances x. That is, the certificate for P and s is:

$$\Pi_s := \min_x \Pi_s(x) = \min_x x(1+\alpha_s(x)) \ .$$

Note that, if not the whole boundary is visible from s, the radial maximal distance function can have incontinuouities. This means that in some directions there is more than one boundary point and the maximal distance *jumps*. To be on the safe side we do not change the definition and simply add a vertical segment into the radial distance function between these *jumping* maximal distances; see for example Figure 6.13 in the Appendix **??**. The corresponding points of the segment lie outside the polygon and the certificate guarantees to leave the polygon.

Of course, the escape path definition could also be changed in order to take the distance to *all* boundary points into account. The radial maximal distance computation would be no longer a function in this case, it is rather a curve. This would also mean that an online escape path has to cope with environments that are gashed like the escape path. We would like to keep the definition as simple as possible and we precisely handle all cases where the starting point is in *star-shaped position*.

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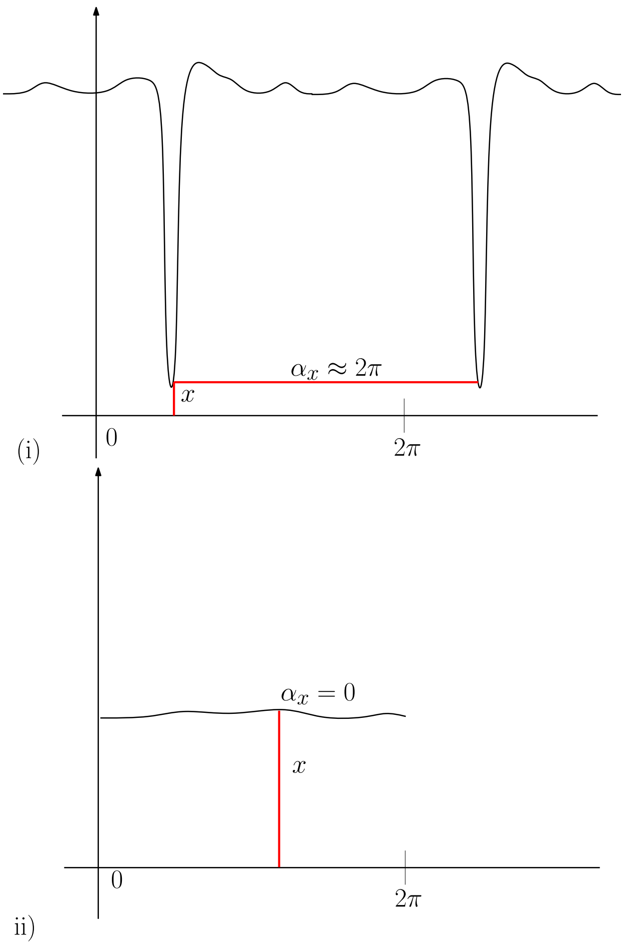

Figure 6.11: A sketch of the radial distance curves of the environments in Figure 6.10 for the origin s. The value $x(1+\alpha_x)$ is the minimal length so that the two segments of length x and α_x hits the boundary of the distance curve for any starting angle of the segment x. In each figure the worst case starting direction for the segment x is shown. The length $x(1+\alpha_x)$ in the polygonal setting corresponds to the volume below the segment α_x plus the distance x.

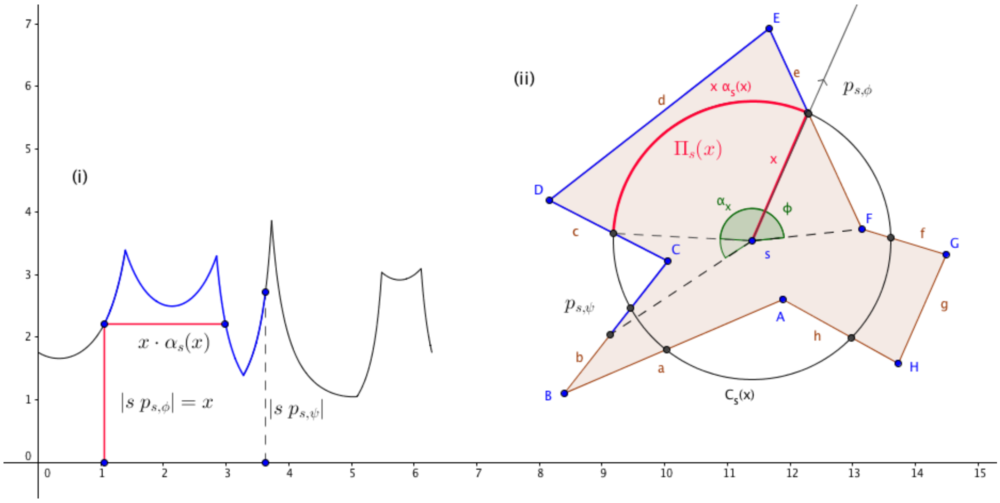

Figure 6.12: (ii) Consider the polygon P and a starting point s. Let us assume that we radially sweep the boundary of P (starting from point F with angle 0) in counterclockwise order and calculate the maximal distance from the boundary to s for any angle. (i) shows this *radial maximal distance function* of the boundary of P from s in polar coordinates for the interval $[0, 2\pi]$. The blue sub-curve corresponds to the blue boundary part in (ii). The certificate path $\Pi_s(x)$ for distance x is the longest path that successfully checks the distance x by a circular strategy. This means that it hits the boundary for any starting direction ϕ of x in P. In the polar-coordinate setting in (i) this is a path with two line segments of length x and α_x that always hits the boundary of the radial distance function independent from the starting angle ϕ. For the case where the boundary of P is not totally visible an example is given in Figure 6.13.

corresponding points of the segment lie outside the polygon and the certificate guarantees to leave the polygon.

6.3.2 Interpretation of the certificate for a maximal distance distribution

In general the definition of a certificate path takes the maximal distances from the starting point s into account. If only the *distribution* of the maximal distances of an arbitrary environment E is known, we can define a polygon that represents this *distribution*; see Figure 6.14. The certificate for this polygon is also an escape path for E. This interpretation resembles the discrete variant of Kirkpatrick; compare to Figure 6.7(iv) in Section 6.2.

We can always define an environment that represents a distance *distribution* of maximal distances as represented in Figure 6.14. Here the boundary of the polygon P is defined in polar coordinates by $(\phi, x(\phi))$ for $\varphi \in [0, 2\pi)$ where $x(\phi)$ is an increasing function (say the distance distribution for some E.). The environment is made polygonal by a line segment l from $A = (0, x(0))$ to a point $B = (2\pi, x(2\pi))$.

In this case the certificate path $\Pi_s(x(\phi))$ for distance $x(\phi)$ is given by a segment s to $(\phi, x(\phi))$ and the clockwise circular arc of angle $(2\pi - \phi)$ that hits l; see Figure 6.14. Thus, the certificate path $\Pi_s(x(\phi))$ represents the *maximal amount* of distances that are larger than $x(\phi)$ all others are smaller than or equal to $x(\phi)$. If we move along $\Pi_s(x(\phi))$ starting from s in an arbitrary direction we will finally always met a boundary point of P at distance $\leq x(\phi)$.

Note that alternatively, the certificate path $\Pi_s(x(\phi))$ can start with a segment of length $x(\phi)$ almost touching the segment AB and moving *counterclockwise* along the arc. For the overall certificate path the angle ϕ is chosen so that $x(\phi)(1 + 2\pi - \phi)$ is minimal among all $\phi \in [0, 2\pi]$.

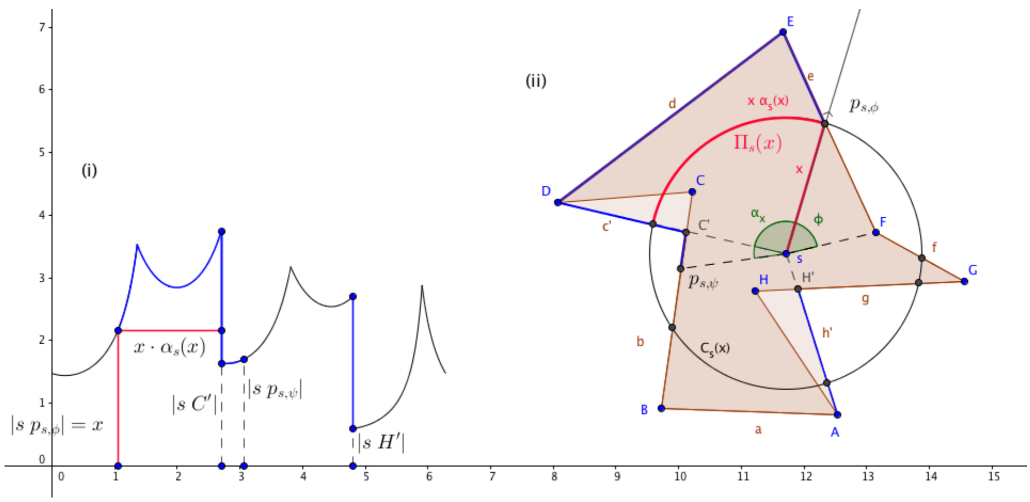

Figure 6.13: (ii) If parts of the boundary of polygon $P = ABCDEFGH$ are not visible from s the *maximal* distance to the boundary jumps for example from sD to sC'. The points in between lie outside the polygon, the certificate is a bit pessimistic but we are on the safe side. We make use of a vertical segment in the maximal radial distance function in (i). In principle we make use of the radial maximal distance function of a polygon P' that fully contains P and the boundary of P' is visible from the staring point. Then we define the certificate for P' wich is $ABC'DEFGH'$ in the above case.

6.3.3 The certificate outperforms the best known escape paths

The best escape path as considered by Bellman and others was designed as a single deterministic finite path for all starting positions of a given fixed environment. It leads out of the given environment from any starting position. There is a worst case position where the full path length is required, the environment is left with the very last point. The monograph of Finch and Wetzel (2005) gives a nice overview.

If the escape path is given, for any starting point the adversary can rotate the environment so that the path length for leaving the environment with this path is maximal. The same holds for the certificate path computed for any given starting point, the worst case is simply given by the length of the certificate. So we can compare the two worst case rotations for any starting point.

For circles, semi-circles with an opening angle α larger than 60 degrees and for *fat* convex bodies it was proved that the diameter is the best escape path; see the examples shown in Figure 6.16 where also the definition of fatness is given.

This means that for an arbitray position the agent moves along a line segment. For any position s the worst case for this escape path is given by a rotation of the environment so that a segment of maximal length x has to be used. Our certificate will use such a path as a possible alternative $\Pi_s(x)$ of distance x with angle $\alpha_x = 0$. This means that in all environments where the diameter is the best escape path, the certificate is as least as good as this path for any position. Note, that the certificate will definitely be better for many starting points.

Certificate versus Besicovitchs path

Consider an equilatateral triangle as depicted in Figure 6.15. Besicovitch Zig-Zag path is the up to now best known escape path for this problem. The Zig-Zag path is symmetric and consist of three segments of length $\sqrt{21}/14$ which gives total length ≈ 0.981918; see also Section 6.1.2.

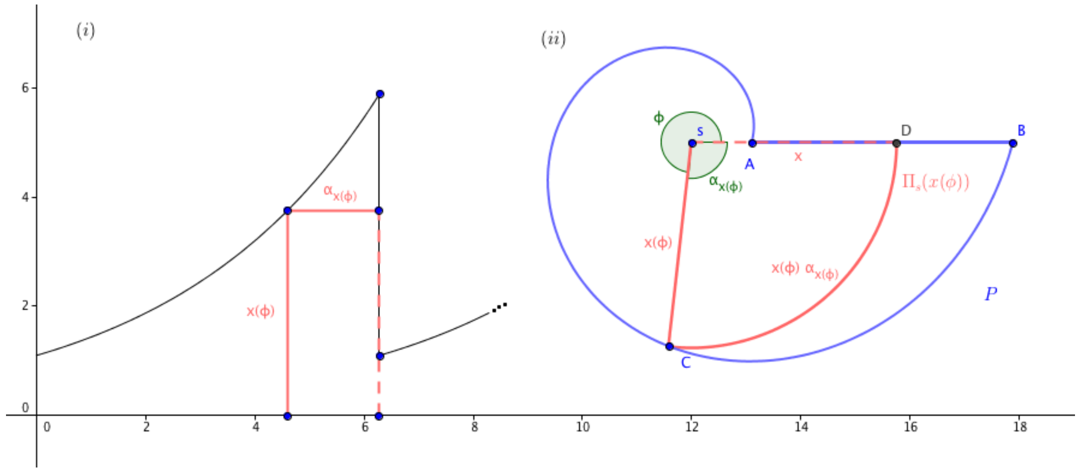

Figure 6.14: An environment P in (ii) where the radial distance curve in (i) is an increasing function $(\phi, x(\phi))$ (except for the closing segment). The certificate path $\Pi_s(x(\phi))$ consists of a segment of length $x(\phi)$ and an arc of length $x(\phi) \cdot \alpha_{x(\phi)}$ with $\alpha_{x(\phi)} = 2\pi - \phi$. Note that $\Pi_s(x(\phi))$ can also start with a segment almost parallel with the X-axis as depicted by the dashed segments in (i) and (ii). The certificate path represents the *maximal amount* of distances that are larger than $x(\phi)$.

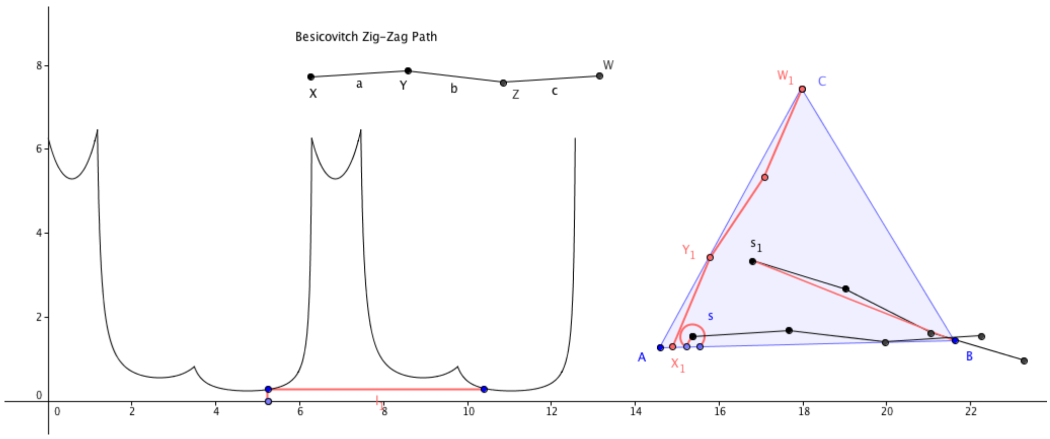

Figure 6.15: The Zig-Zag path of Besicovitch is assumed to be the best escape path for an equilateral triangle. A worst case position is for example given by X_1. On the lefthandside the radial distance curve of point s is shown. The certificate path for such points close to the boundary is very short. The usage of the Zig-Zag path is much worser. If the starting point s_1 is a bit away from the boundary the Zig-Zag path attains a worst case for a leaving point on the boundary farthest away. Thus, the certificate is also always better in such cases.

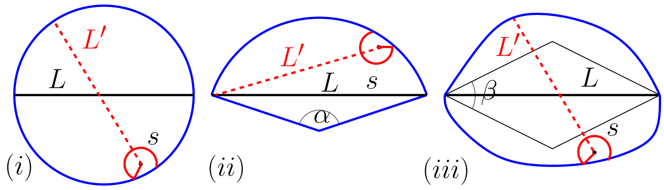

Figure 6.16: Three environments where the diameter L is the best escape path. (i) A circle, (ii) a semi-circle with angle $\alpha \geq 60°$. (iii) A so-called *fat* convex body, where fatness is defined over a rhombus with angle β at least $60°$, the diameter of the rhombus equals the diameter of the body and the rhombus has to fit into the environment. Using the optimal escape path (line segment L) from an arbitrary starting point s results in the largest distance x to the boundary (dashed path L') in the worst case. This is always also the certificate path $\Pi_s(x)$ of distance x with $\alpha_x = 0$. So the minimal certificate path is as least as good as the diameter path.

An example of a worst case starting point X_1 is given in Figure 6.15.

For starting points somewhere in the center of the triangle, the Zig-Zag path is worse than the largest distance to the boundary; see for example starting point s_1 in Figure 6.15. Thus, the certificate is shorter for these points. For starting points close to the boundary the certificate is significantly better by a short circular check; see for example starting point s_1 in Figure 6.15. We sketch a formal proof, that the Zig-Zag path can never beat the certificate, the angular values are chosen from the optimization of Besicovitch path from Section 6.1.2. Thus, in Figure 6.17 we have $\alpha = \arcsin\left(\frac{1}{\sqrt{28}}\right) \approx 10.9°$ and $x = \sqrt{3/28}$.

If the Besicovitch Zig-Zag path does not hit a point on the boundary with maximum distance away from the starting point, it does not hit one of the three vertices of the triangle. Only in this case Besicovitch Zig-Zag path can beat the certificate. In this case the Besicovitch path has to make use of at least two segments (each of length $x = \sqrt{3/28}$) for leaving the triangle. As shown in Figure 6.17 only a small area close to a vertex of the triangle has to be checked. For all those points we find a circular strategy that is shorter than $2x$ which is always required by the Zig-Zag path.

6.4 Online approximation of the certificate path

We are searching for a short escape path in an unknown environment. As shown in the previous section the certificate path and its length is a reasonable candidate for comparisons.

We apply the following logarithmic spiral strategy. A logarithmic spiral can be defined by polar coordinates $(\varphi, a \cdot e^{\varphi \cot(\alpha)})$ for $\varphi \in (-\infty, \infty)$, a constant a and an excentricity β as shown in Figure 6.18. For an angle ϕ the path length of the spiral up to point $(\phi, a \cdot e^{\phi \cot(\beta)})$ is given by $\frac{a}{\cos\alpha} e^{\phi \cot(\beta)}$. For our purpose we choose β so that the two extreme cases of the certificate attain the same ratio; see Figure 6.18. We can assume that the certificate of the environment is $x(1+\alpha_x)$ for an arbitrary distance x and an angle $\alpha_x \in [0, 2\pi]$. Since the spiral strategy checks the distances in a monotonically increasing and periodical way, there has to be some angle ϕ so that $x = a \cdot e^{(\phi-\alpha_x)\cot(\beta)}$ holds. This means that in the worst case the spiral strategy will leave the environment at point $p = (\phi, a \cdot e^{\phi \cot(\beta)})$ with path length $\frac{a}{\cos\beta} \cdot e^{\phi \cot(\beta)}$. Exactly α_x distances of length x have been exceeded, which means that the boundary has to be visited.

We would like to choose β so that the two extreme cases $\alpha_x = 0$ and $\alpha_x = 2\pi$ have the same

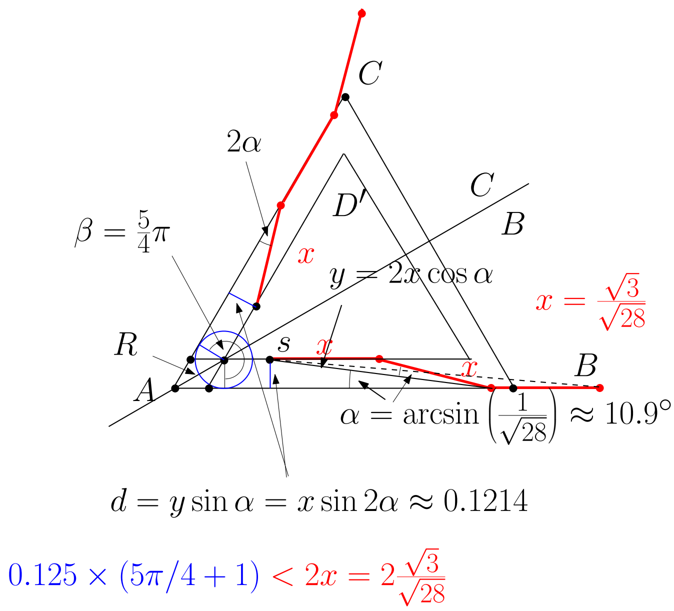

Figure 6.17: For comparing the certificate and the Zig-Zag path we only have to consider the case where the triangle cannot be rotated so that the Besicovitsch path ends in the vertex farthest away. For starting points s below the bisector of A and C this can only happen if s is at least vertical distance $d = x \sin 2\alpha$ away from segment AC. For starting points above the the bisector of A and C this can only happen, if s is at least vertical distance $d = x \sin 2\alpha$ away from segment AB. For both cases it remains to consider the points in the rhomboid R. Fortunately, we can use a circle of radius $d' = 0.125$ (slightly larger than d) so that the circle of radius d' with starting point in R touches the boundary with an arc of lenght at most $2\pi - \frac{2}{3}\pi = \frac{4}{3}\pi$ and $d'(\frac{4}{3}\pi + 1)$ is always strictly smaller than $2x$.

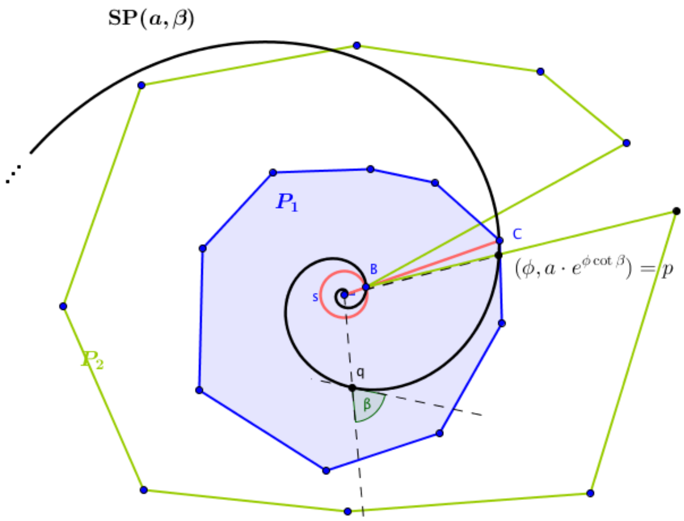

Figure 6.18: We apply a spiral strategy at the starting point s for arbitrary unknown polygons. The excentricity β is chosen so that the two extreme cases always have the same ratio. Here for both polygons P_1 and P_2 in the worst case the strategy passes the boundary at point $p = (\phi, a \cdot e^{\phi \cot \beta})$ close to C. The path length of the strategy for leaving the polygons is roughly the same. The certificate for P_1 has length $|s\, C|$ (checking the maximal distance to the boundary of P_1) whereas the certificate for P_2 has length $|s\, B|(1 + 2\pi)$ with $|s\, C| = e^{2\pi \cot \beta}|s\, B|$ (checking the smallest distance to the boundary of P_2 with a full circle). We can construct such examples for any point p on the spiral.

ratio. Thus, we are searching for an angle β so that

$$\frac{\frac{a}{\cos\beta} \cdot e^{\phi \cot \beta}}{a \cdot e^{\phi \cot \beta}(1+0)} = \frac{\frac{a}{\cos\beta} \cdot e^{\phi \cot \beta}}{a \cdot e^{(\phi - 2\pi) \cot \beta}(1+2\pi)} \Leftrightarrow \tag{6.1}$$

$$1 = \frac{e^{2\pi \cot \beta}}{1 + 2\pi} \tag{6.2}$$

holds. The righthandside of Equation (6.1) shows the case where $x_2 = a \cdot e^{(\phi - 2\pi)\cot(\beta)}$ and $\alpha_{x_2} = 2\pi$ gives the certificate and the lefthandside shows the case that $x_1 = a \cdot e^{\phi \cot(\beta)}$ and $\alpha_{x_1} = 0$ gives the certificate $x_i(1 + \alpha_{x_i})$, respectively. In both cases the spiral will detect the boundary latest at point $p = (\phi, a \cdot e^{\phi \cot(\alpha)})$ because the spiral checks 2π distances larger than or equal to x_2 and at least one distance x_1. Figure 6.18 shows the construction of corresponding polygons P_1 and P_2.

The solution of Equation (6.2) gives $\beta = \operatorname{arccot}\left(\frac{\ln(2\pi+1)}{2\pi}\right) = 1.264714\ldots$ and the ratio is $\frac{1}{\cos\beta} = 3.3186738\ldots$. Fortunately, for all other values $x = a \cdot e^{(\phi-\gamma)\cot\beta}$ and $\alpha_x = \gamma$ for $\gamma \in (0, 2\pi)$ the ratio is smaller than these two extremes. The overall ratio function is

$$f(\gamma) = \frac{\frac{a}{\cos\beta} \cdot e^{\phi \cot \beta}}{a \cdot e^{(\phi-\gamma)\cot\beta}(1+\gamma)} = \frac{e^{\gamma \cot \beta}}{\cos\beta(1+\gamma)} \text{ for } \gamma \in [0, 2\pi] \tag{6.3}$$

and Figure 6.19 shows the plot of all possible ratios of the spiral strategy with excentricity β. Altogether, we have the following result.

Theorem 76 *There is a spiral strategy for any unknown starting point s in any unknown environment P that approximates the certificate for s and P within a ratio of* 3.31864.

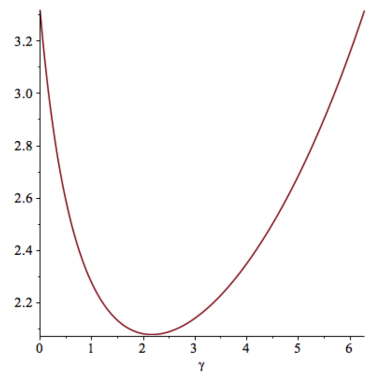

Figure 6.19: The graph of the ratio function f of Equation (6.3) for the spiral strategy with excentricity $\beta \approx 1.26471$. The two extreme cases 0 and 2π have the same ratio ≈ 3.31864 and all other ratios are strictly smaller.

Proof.

Assume that the certificate of P and s is given by $x(1+\alpha_x)$. We can set $\gamma := \alpha_x$ and we will also find an angle ϕ so that $x = a \cdot e^{(\phi-\gamma)\cot\beta}$ holds. At point $p = (\phi, a \cdot e^{\phi \cot \beta})$ the spiral has subsumed an arc of angle γ with distances x, so the spiral strategy will leave P at p in the worst case. The ratio is given by $f(\gamma)$ as in (6.3) and Figure 6.19. In the worst case for the strategy γ is either 0 or 2π for the ratio 3.31864, respectively. □

Note that for the starting situation we can start the spiral after a movement along a segment of fixed length a. Such starting problems are covered by an additive constant in the competitive analysis framework.