Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Geometric Firefighting – Lower Bound and FF Curve

Elmar Langetepe

University of Bonn

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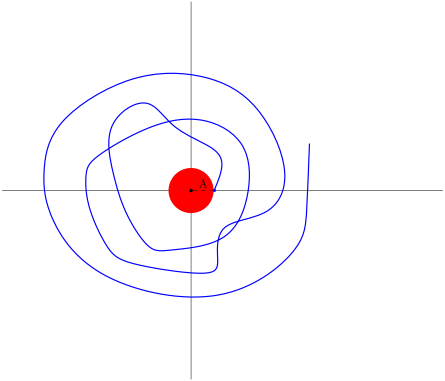
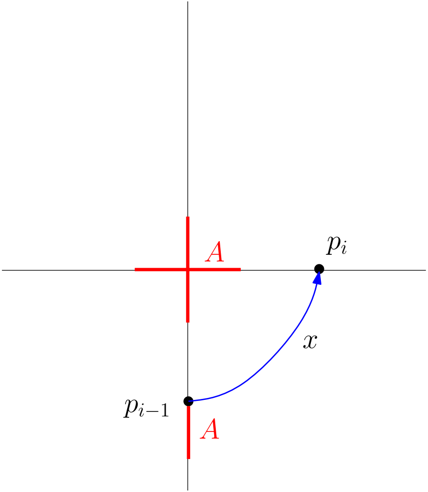
Lower bound construction, spiralling strategies!

- Start at the fire!
- Spiralling strategies!
- Visit four axes in cyclic order
- Visit axes in increasing distance



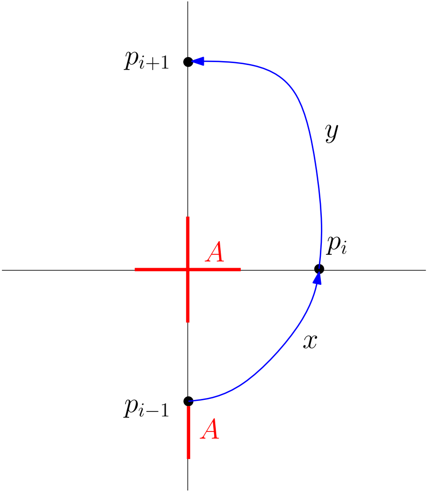
Theorem 58: Each "spiralling" strategy must have speed $v > 1.618\ldots$ (golden ratio) to be successful.

Proof of lower speed bound: suppose $v \leq 1.618$



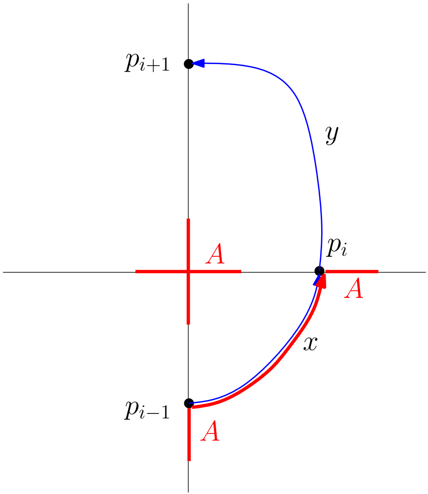
By induction:
On reaching p_i,
interval of length A below
p_{i-1} is on fire.

(Induction base!)



Inductive Step:

After arriving p_{i+1}
fire moves at least $x + A$



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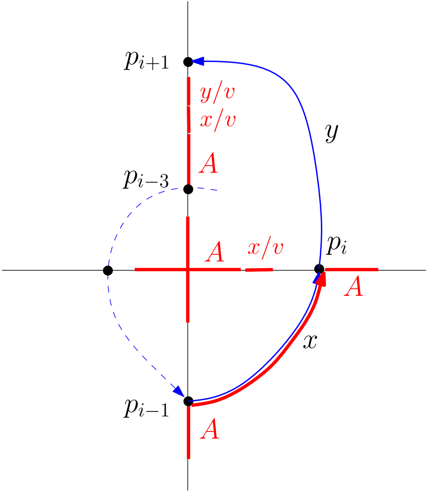


On reaching p_{i+1}:

1. $A + \frac{x}{v} \leq p_i \leq x$ and
2. $A + \frac{x}{v} + \frac{y}{v} \leq p_{i+1} \leq y$

$\Longrightarrow \frac{1}{v(v-1)} x + \frac{1}{v-1} A \leq \frac{y}{v}$

$\Longrightarrow x + A \leq \frac{y}{v}$

from $v^2 - v \leq 1$

FollowFire Strategy for $v = 5.27$!

Logarithmic spiral of excentricity α around Z ($\frac{1}{v} = \cos(\alpha)$)!

(First Part)

FollowFire Strategy for $v = 5.27$!

Logarithmic spiral of excentricity α around p_0 ($\frac{1}{v} = \cos(\alpha)$)!

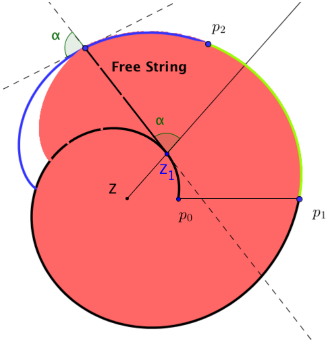
(Second Part)

Excentricity α around wrapping center Z_1 ($\frac{1}{v} = \cos(\alpha)$)!

(Third part!)

FollowFire: Free String Wrapping!

- $v = 5.27$ ($\alpha = 1.38$)
- $\text{Log}(\mathbf{p_0}, \mathbf{p_1})$, $\text{Log}(\mathbf{p_1}, \mathbf{p_2})$
- Free string: $F_1(l)$:
  Wrapping around $\text{Log}(\mathbf{p_0}, \mathbf{p_1})$



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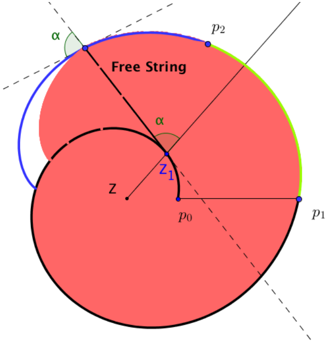
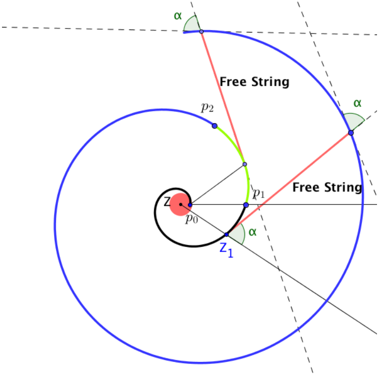


- $v = 3.07$ ($\alpha = 1.24$)
- Wrapping around $\mathrm{Log}(\mathbf{p_1}, \mathbf{p_2})$



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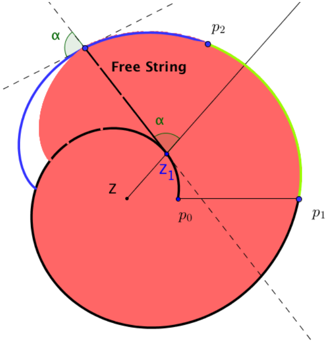

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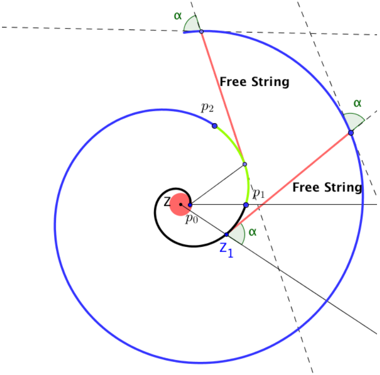
Wrapping around wrappings!



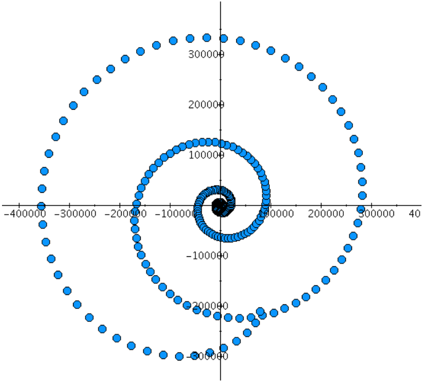
Experimental approach!

(Spiral Generator Appet!)

FollowFire: Successful?

$v = 2.69$ ($\alpha = 1.19$):
8 rounds!

$v = 2.593$ ($\alpha = 1.175$):
Simulation did not succeed!



Successful for which $v \in (1, \infty)$?

Lower and upper bounds on v! Proofs!

Upper bound by FollowFire

Theorem 59: FollowFire strategy is successful if $v > v_c \approx 2.6144$

Upper bound by FollowFire

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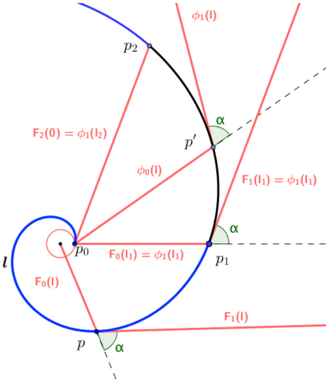
Sketch! When gets the free string to zero?

1. Parameterize free strings for coil j (Linkage)
2. Structural properties
3. Successive interacting differential equations
4. Inserting end of parameter interval
5. Coefficients of power series
6. Ph. Flajolet: Singularities
7. Pringsheim's Theorem and Cauchy's Residue Theorem

Upper bound: 1. Parameterize the free string

FollowFire Wrapping process!

Free strings F_j/ϕ_j parameterized by lenght of starting spirals!



$$|Log(p_0, p_1)| = l_1$$
$$|Log(p_0, p_1)| + |Log(p_1, p_2)| = l_2$$

$$F_j\colon\ l \in [0, l_1]$$
$$\phi_j\colon\ l \in [l_1, l_2]$$

Upper bound: 1. Parameterize the free string (Linkage)

FollowFire Drawing backwards tagents!

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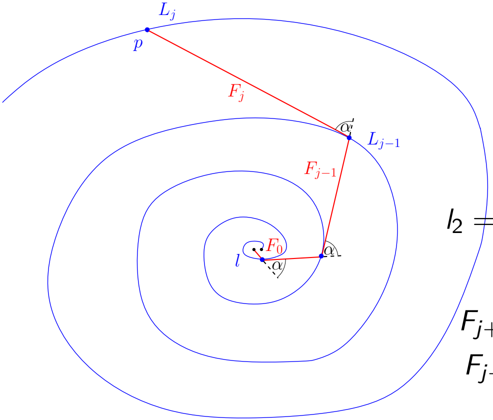

F_j: $l \in [0, l_1]$
ϕ_j: $l \in [l_1, l_2]$

$$l_1 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1)$$

$$l_2 = \frac{A}{\cos\alpha}(e^{2\pi \cot\alpha} - 1)e^{\alpha \cot\alpha}$$

$$F_{j+1}(l_1) = \phi_{j+1}(l_1)$$
$$F_{j+1}(0) = \phi_j(l_2)$$

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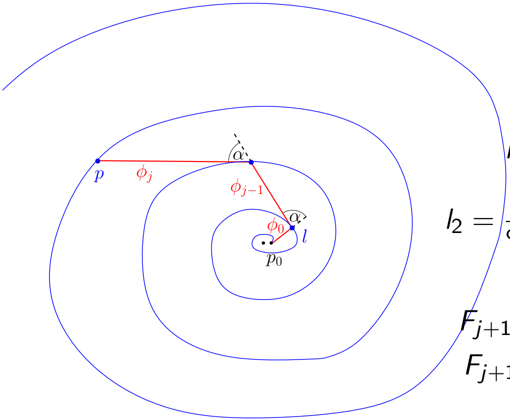

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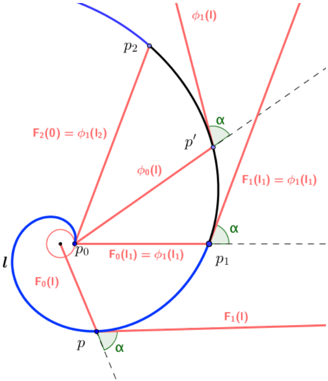

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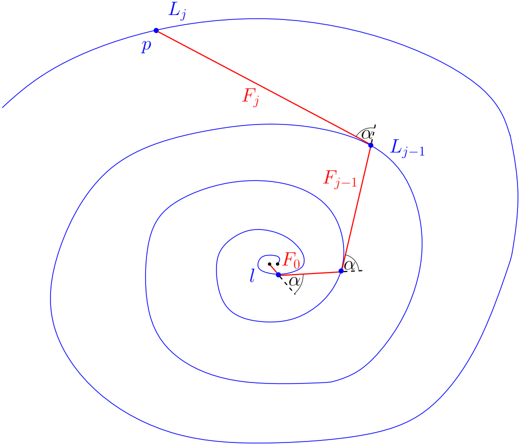
$$F_j\colon\ l \in [0, l_1]$$
$$\phi_j\colon\ l \in [l_1, l_2]$$

$$\begin{aligned} F_{j+1}(l_1) &= \phi_{j+1}(l_1) \\ F_{j+1}(0) &= \phi_j(l_2) \\ F_0(l) &= A + \cos(\alpha)\, l \end{aligned}$$

Parameterized by lenght l of starting spirals!

$L_j(l)$ length of the curve! $F_j(l)$ (and $\phi_j(l)$) length of the free string!



Lemma 60:

$$L_{j-1} + F_j = \cos\alpha\, L_j$$

Lemma 61:

$$\frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}}$$

Helping Lemmata

Lemma 60: $L_{j-1} + F_j = \cos\alpha\, L_j$

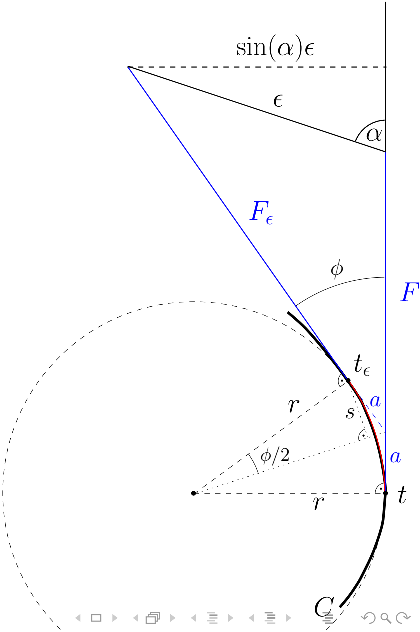
- Fire and fire fighter, reach endpoint at $F_j(l)$ at the same time
- Unit-speed fire, geodesic distance of $L_{j-1}(l) + F_j(l)$
- Fighter distance of $L_j(l)$ at speed $1/\cos\alpha$

Lemma 61: $\frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}}$

Lemma 62: String of length F is tangent to point t on smooth curve C. End of string moves distance ϵ in direction α. For the curve length $C_t^{t_\epsilon}$ between t and the new tangent point, t_ϵ, we have

$$\lim_{\epsilon \to 0} \frac{C_t^{t_\epsilon}}{\epsilon} = \frac{r \sin \alpha}{F}$$

where r denotes radius of osculating circle at t.



$r\ \sin(\phi/2) = s = a\ \cos(\phi/2)$
gives $a = r\ \tan(\phi/2)$

$2a$ approximates $c := C_t^{t_\epsilon}$:

$$\frac{c}{2a} = \frac{r\,\phi}{2r\ \tan(\phi/2)} \approx \cos^2(\phi/2) \to 1$$

$$\frac{\epsilon\ \sin(\alpha)}{\sin(\phi)} = \frac{F_\epsilon + a}{\sin(\pi/2)} \text{ gives } \frac{\sin(\phi)}{\epsilon} = \frac{\sin(\alpha)}{F_\epsilon + a} \to \frac{\sin(\alpha)}{F}$$

$\sin(\phi/2)/\epsilon \ \to\ \sin(\alpha)/(2F)$

$$\frac{C_t^{t_\epsilon}}{\epsilon} = \frac{c}{2a}\,\frac{2a}{\epsilon} \approx \frac{2r\ \tan(\phi/2)}{\epsilon} = \frac{2r\ \sin(\phi/2)}{\epsilon\ \cos(\phi/2)} \to \frac{r\sin(\alpha)}{F}.$$