Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16

Example Queries for the oral exams

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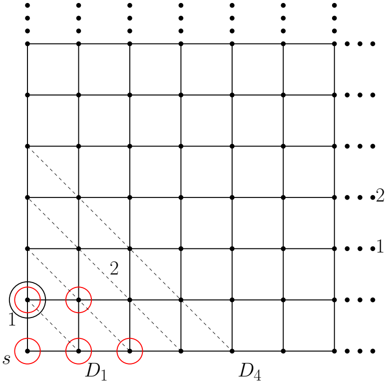
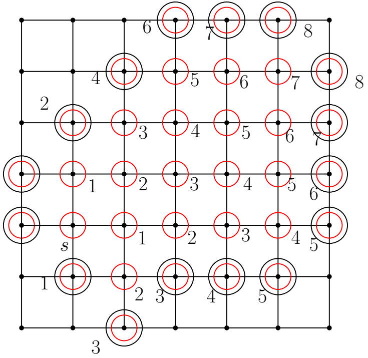
General remarks!

- Repetition of the main statements:
  Problem Def./Theorem/Lemmata
- Top-Down! Proof ideas and details!
- Explanation on examples! Algorithm/Lower Bound!
- Example Questions!
- Not all details are on the foils!
- First questions Q1/Q2 in detail!
- Walk-Through!

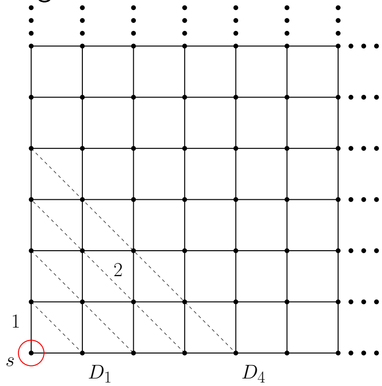
- Model: Grid environment, static variant, moving agent
- Q: How many agents are required?
- Q1: Lower bound, proof in detail
- Q2: Upper bound, proof idea





Q1: Proof detail, Lower Bound $k = 1$

Lemma 2: Catching an evader in a grid world by setting $k = 1$ blocking cells after each movement of the evader cannot succeed in general.



Step l: r_l blocked cells in $D_{l+1}, D_{l+2}, \ldots$
$B_l \subseteq D_l$ burning cells in D_l

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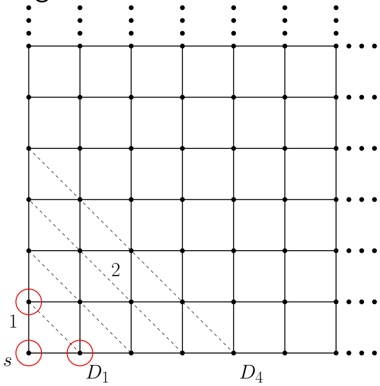

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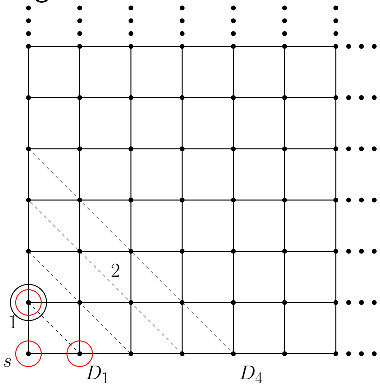

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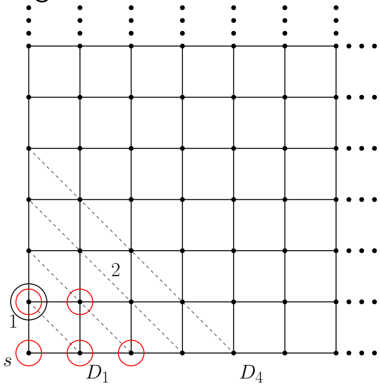

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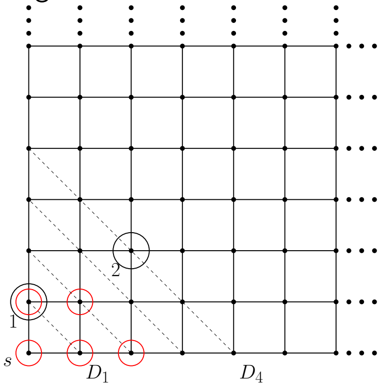

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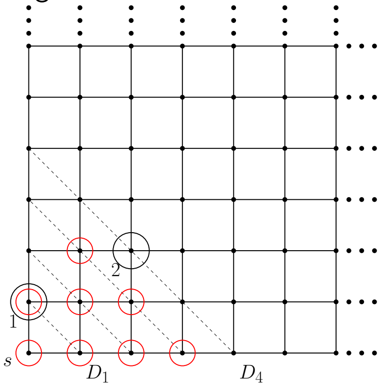

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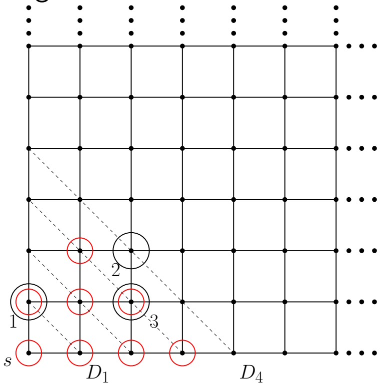

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- Block of the guard in D_{l_1}: $l_1 \leq l + 1$
  $\Rightarrow r_{l+1} = r_l - x$, $B_{l+1} \geq 1 + r_{l+1}$
- Block of the guard in D_{l_1}: $l_1 > l + 1$
  $\Rightarrow r_{l+1} = r_l - x + 1$, $B_{l+1} \geq 1 + r_{l+1}$

Q2. Proof idea, Upper bound! $k = 2$

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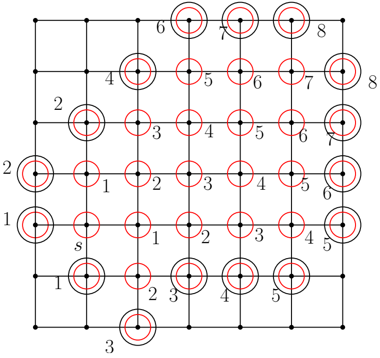

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Firefigthing interpretation! Outside the fire!

Lemma 3: For the outbreak of a fire on a single source in a grid and the usage of two firefighters per time step any successul strategy encloses an area of at least 18 burning vertices. This bound is tight.

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- $d_{v,t} = \begin{cases} 1 & : \quad \text{vertex } v \in L \text{ is defended before or at time } t \\ 0 & : \quad \text{otherwise} \end{cases}$

Q2: Proof idea, Upper bound! $k = 2$

Firefigthing interpretation! Integer LP for $l \leq 8$, $T \leq 9$

$$\text{Min } \sum_{v \in L} b_{v,T}$$

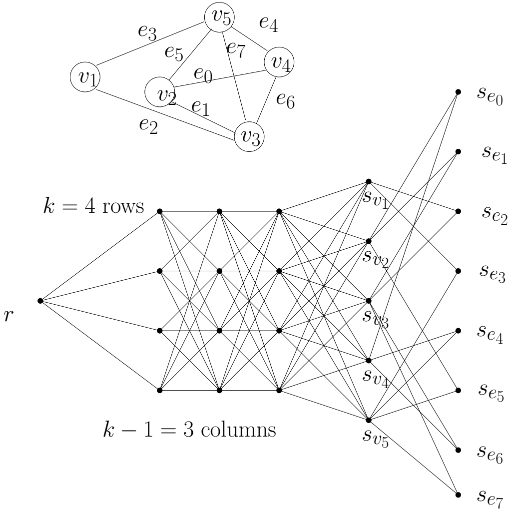
$$
\begin{array}{rcll}
b_{v,t} + d_{v,t} - b_{w,t-1} & \geq & 0 & : \forall v \in L, v \in N(w), 1 \leq t \leq T \\
b_{v,t} + d_{v,t} & \leq & 1 & : \forall v \in L, 1 \leq t \leq T \\
b_{v,t} - b_{v,t-1} & \geq & 0 & : \forall v \in L, 1 \leq t \leq T \\
d_{v,t} - d_{v,t-1} & \geq & 0 & : \forall v \in L, 1 \leq t \leq T \\
\sum_{v \in L}(d_{v,t} - d_{v,t-1}) & \geq & 2 & : \forall 1 \leq t \leq T \\
b_{v,0} & = & 1 & : v \in L \text{ is the origin } (0,0) \\
b_{v,0} & = & 0 & : v \in L \text{ is not the origin } (0,0) \\
d_{v,0} & = & 0 & : \forall v \in L \\
d_{v,t}, b_{v,t} & \in & \{0,1\} & : \forall v \in L, 1 \leq t \leq T
\end{array}
$$

Graphs and Trees

- Same Model: static variant, moving agent, general graph
- How many agents are required?
- Q: Complexity of the problem?
- Q3: Explain the NP-hardness, present reduction in detail
- Q: Polynomial time in some cases?
  - Q: Special graphs?
  - Q: Greedy approximation for trees: Factor and proof!
  - Q: Dynamic programming approach for trees! Explain!

Q3: Static general Graph, Reduction detail: k-Clique

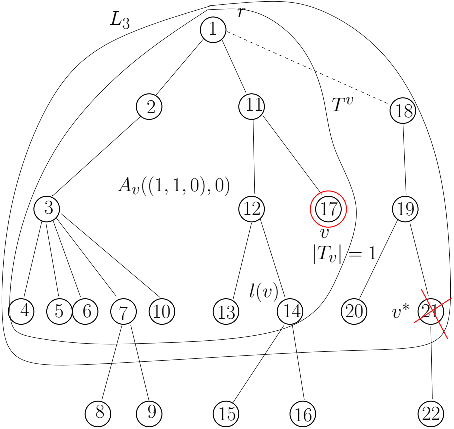
Theorem 10: Firefighter decision problem in graphs: NP-hard.



k-Clique and $k' = k + \binom{k}{2} + 1$ protected vertices

Trees, simple algorithms

- Static: Approximation Greedy, Dynamic Programming (exact)
- Q4: Advantage for trees? Dynammic Programming! Idea!



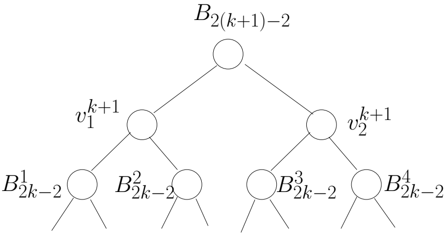
$|T_v| + A_{l(v)}((1,0,0),1)$ or $A_{l(v)}((1,1,0),0)$

Dynamic configuration, structure!

1. Place a team of p guards on a vertex.
2. Move a team of m guards along an edge.

(3. Remove a team of q guards from a vertex)

- Contiguous search (1.+2.) number: $\mathtt{cs}(T) \leq k$
- Theorem 17: Monotone contiguous strategy with all $\mathtt{cs}(T)$ agents that starts in a single vertex.
- Corollary 33: Tree T exists with $cs(T) \leq 2s(T) - 2$.
- Q5 Definition: Progr. connected crusades, frontier at most k
  Q6 Proof idea: Progr. Conn. Crusades frontier k, T and T'
- Q7 Rule 3. What is the difference? Jumping! $\mathtt{cs}(T)$ vs. $s(T)$



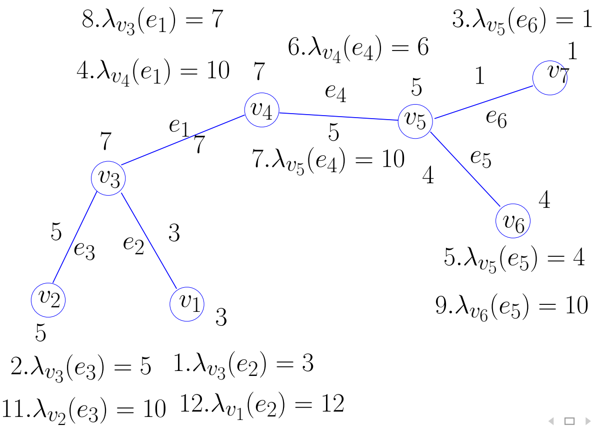
Dynamic configuration, trees, strategy

- Message sending algorithm! Q8 Explain the idea! Analysis!
- Correct only for unit weights! Q9 Explain the problem!

$\mu(v_3) = \max(\lambda_{v_3}(e_1), \lambda_{v_3}(e_3) + 7) = 12$

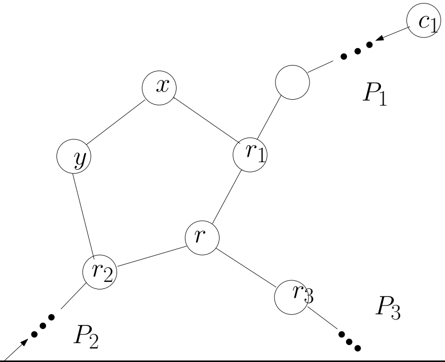
$\mu(v_5) = \max(\lambda_{v_5}(e_4), \lambda_{v_5}(e_5) + 5) = 10$

$10.\lambda_{v_7}(e_6) = 10$



Cop and Robber Problems

- Structural properties: Pitfalls, Classification, If-and-only-if!
  Q10 Explain the concepts/definitions!
- Number of cops required! $c(G)$
- Theorem 41: G max. degree 3, any two adjacent edges are contained in a cycle of length at most 5: $c(G) \leq 3$.
- Theorem 43: For planar graphs: $c(G) \leq 3$
- Q11/12: Explain the proof ideas!



Randomization: Tree, static!

- Greedy approximation: $\frac{1}{2}$, Expetcted: $1 - \frac{1}{e}$
- Q13: Explain the idea, sketch the analysis!

$$\begin{array}{llcll}
\text{Minimize} & \sum_{v \in V} x_v w_v & & & \\
\text{so that} & x_r = 0 & = & 0 & \\
 & \sum_{v \leq u} x_v & \leq & 1 & : \quad \text{for every leaf } u \\
 & \sum_{v \in L_i} x_v & \leq & 1 & : \quad \text{for every level } L_i, i \geq 1 \\
 & x_v & \in & \{0, 1\} & : \quad \forall\, v \in V
\end{array}$$

$$\mathbf{Pr}[y_v = 1] = 1 - \prod_{i=1}^{k} (1 - x_{v_i}^F) \geq \left(1 - \frac{1}{e}\right) y_v^F .$$

Randomization: Search number, random fire

- Minimal number k such that proportional part can be safed
- $s_k(G) \geq \epsilon$: $\frac{1}{|V|} \sum_{v \in V} \mathrm{sn}_k(G, v) \geq \epsilon |V|$
- Q14: Explain the definitions!
- Theorem 46: Planar graphs,no 3- and 4-cycle: $s_2(G) \geq 1/22$. Analysis:
  - Let X_2 denote the vertices of degree ≤ 2.
  - Let Y_4 denote the vertices of degree ≥ 4.
  - Let X_3 denote the vertices of degree exactly 3 but with at least one neighbor of degree ≤ 3.
  - Let Y_3 denote the vertices of degree exacly 3 but with all neighbors having degree > 3 (degree 3 vertices not in X_3).
- Q15: Explain the analysis:

$$s_2(G) \geq \frac{n-2}{n} \cdot \frac{x_2 + x_3}{x_2 + x_3 + y_3 + y_4} \geq \frac{n-2}{n} \cdot \frac{x_2 + x_3}{21(x_2 + x_3)} = \frac{n-2}{21n}.$$

- Theorem 1: Computing optimal-enclosement-sequence: NP-hard. (Q: Present Reduction!)

Global Greedy! Q16: Explain the prerequisites/the idea!

- Sort remaining jobs b_j by $\frac{A_j(J_n)}{d_j}$, process largest!

1. b_j can be scheduled somewhere in J_n. Insert b_j: J_{n+1}
2. b_j cannot be processed, overlaps with jobs in J_n.
   Find sequence in J_n that overlaps:
   1. Profits of these jobs smaller than μ times $A_j(J_n)$.
   2. b_j can be scheduled after deletion of the jobs.

   Then build J_{n+1} with b_j. Deleted jobs vanish forever!
3. No such sequence exists in J_n. Reject b_j!

Geometric Fire Fighting: Global Greedy

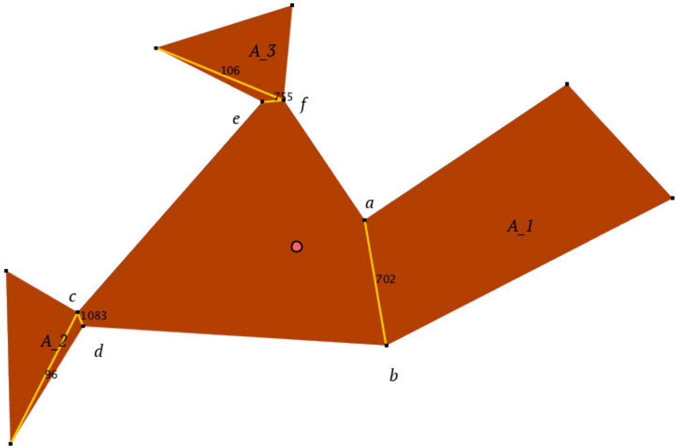
Q16 Explain the analysis in detail!

$$
\begin{aligned}
|J_{\mathrm{opt}}| &\leq J_m(blue) + J_m(green) + J_m(grey) && (1)\\
&\leq \left(2 + \frac{2}{\mu}\right)(J_m(green) + J_m(grey)) && (2)\\
&\leq \frac{2(\mu+1)}{\mu}\left(J_m(green) + \frac{\mu}{1-\mu} J_m(green)\right) && (3)\\
&\leq \frac{2(\mu+1)}{\mu}\frac{1}{1-\mu} J_m(green) && (4)\\
&\leq 2\frac{\mu+1}{\mu(1-\mu)} J_m(green) \leq 2\frac{\mu+1}{\mu(1-\mu)} |J_m| \,. && (5)
\end{aligned}
$$

Explain Inequalities: Grey vs. Green! (3)

Paying scheme: Blue vs. Grey and Green (2) !

Theorem 55: Geometric firefighter problem inside a simple polygon with non-intersecting barriers, approximation algorithms saves at least $\frac{1}{6+4\sqrt{2}} = \frac{3}{2} - \sqrt{2} \approx 0.086$ times area of the optimal solution.

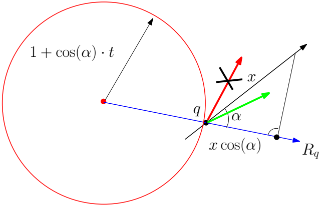


Q17 Example/Problem with intersecting barriers! Explain!

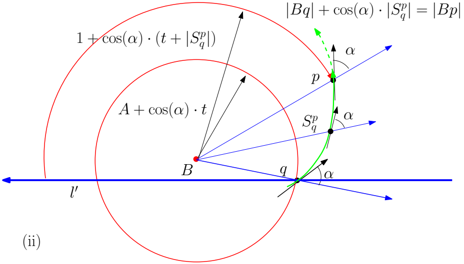
Geometric Fire Fighting: Plane

Spiral strategy is reasonable!

Q18 Explain the relationship: Speed/Spiral!

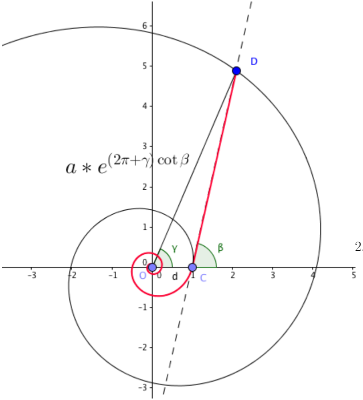


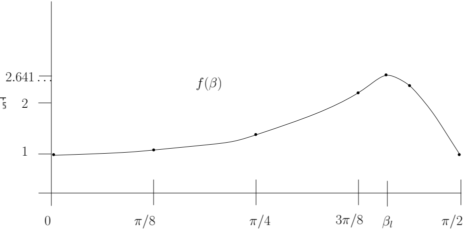
(i)



(ii)

Theorem 56: Speed $v > v_l \approx 2.6145$ success of spiralling strat.





Q19 Explain the Strategy Idea!

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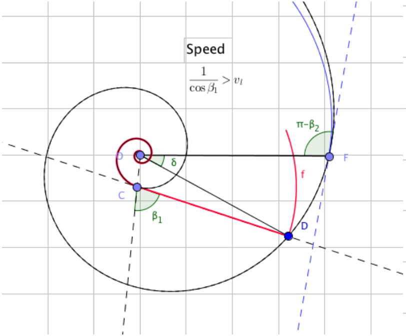
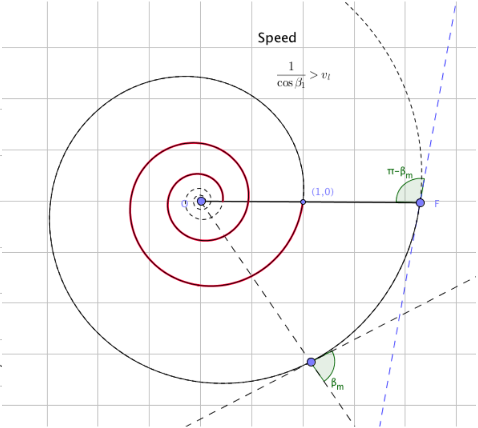
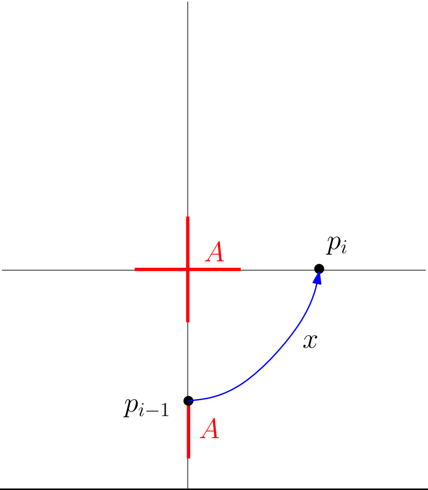




Theorem 58: Successf. spiralling strategy must be of speed $v > \frac{1+\sqrt{5}}{2} \approx 1.618$.

Q19 Explain the Lower Bound constr. in detail!



By induction:
On reaching p_i, interval of length A below p_{i-1} is on fire.

(Induction base!)

Proof of lower speed bound: suppose $v \leq 1.618$

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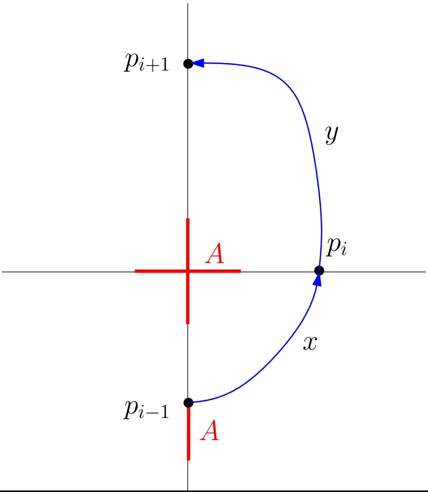

Inductive Step:

After arriving p_{i+1}
fire moves at least $x + A$

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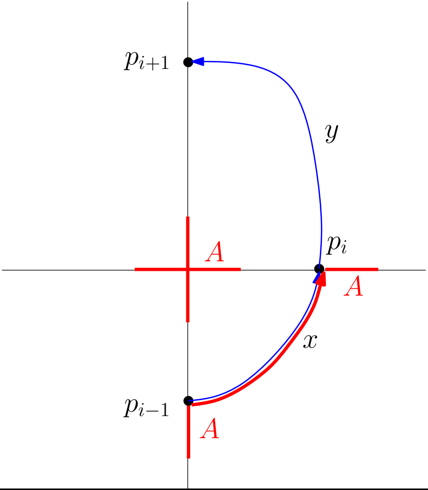

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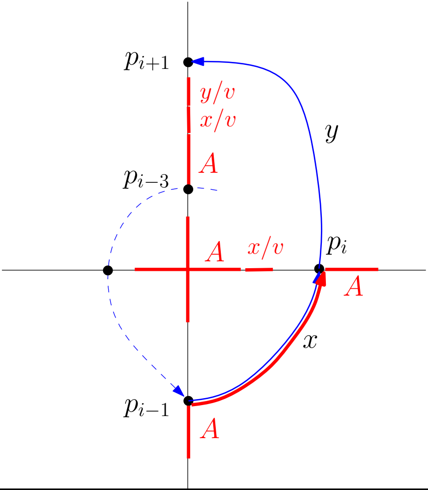


On reaching p_{i+1}:

1. $A + \frac{x}{v} \leq p_i \leq x$ and
2. $A + \frac{x}{v} + \frac{y}{v} \leq p_{i+1} \leq y$

$\Longrightarrow \frac{1}{v(v-1)} x + \frac{1}{v-1} A \leq \frac{y}{v}$

$\Longrightarrow x + A \leq \frac{y}{v}$

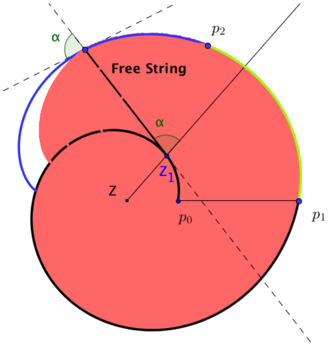
from $v^2 - v \leq 1$

Alternative Strategy FollowFire: Free String Wrapping!

Theorem 59: Strategy FF contains the fire if $v > v_c \approx 2.6144$.

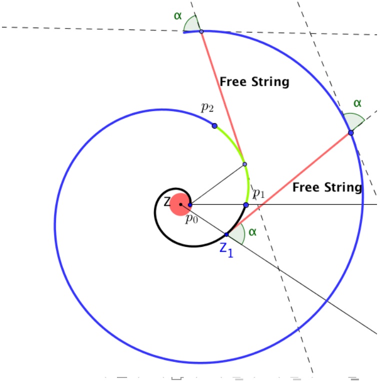
Q20 Explain the idea and sketch the proof!

- $v = 5.27$ ($\alpha = 1.38$)
- $\mathrm{Log}(\mathbf{p_0}, \mathbf{p_1})$, $\mathrm{Log}(\mathbf{p_1}, \mathbf{p_2})$
- Free string: $F_1(l)$:
  Wrapping around $\mathrm{Log}(\mathbf{p_0}, \mathbf{p_1})$



- $v = 3.07$ ($\alpha = 1.24$)
- Wrapping around $\mathrm{Log}(\mathbf{p_1}, \mathbf{p_2})$

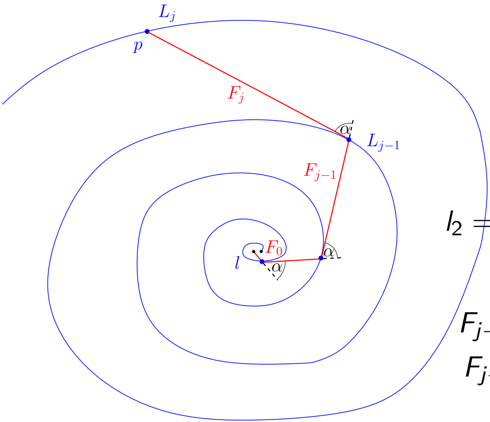
Wrapping around wrappings!



Q20 Explain the idea and sketch the proof!

FollowFire Drawing backwards tangents!

Free strings F_j/ϕ_j parameterized by lenght of starting spirals!



F_j: $l \in [0, l_1]$
ϕ_j: $l \in [l_1, l_2]$

$$l_1 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1)$$

$$l_2 = \frac{A}{\cos\alpha}(e^{2\pi\cot\alpha} - 1)e^{\alpha\cot\alpha}$$

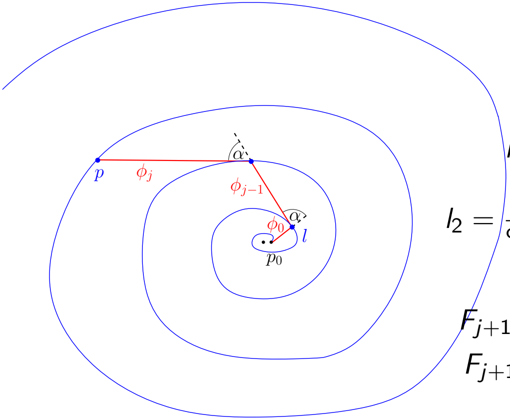
$$\begin{aligned} F_{j+1}(l_1) &= \phi_{j+1}(l_1) \\ F_{j+1}(0) &= \phi_j(l_2) \end{aligned}$$

Upper bound: Parameterize the free string (Linkage)

Q20 Explain the idea and sketch the proof!

FollowFire Drawing backwards tangents!

Free strings F_j/ϕ_j parameterized by lenght of starting spirals!



F_j: $l \in [0, l_1]$
ϕ_j: $l \in [l_1, l_2]$

$$l_1 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1)$$

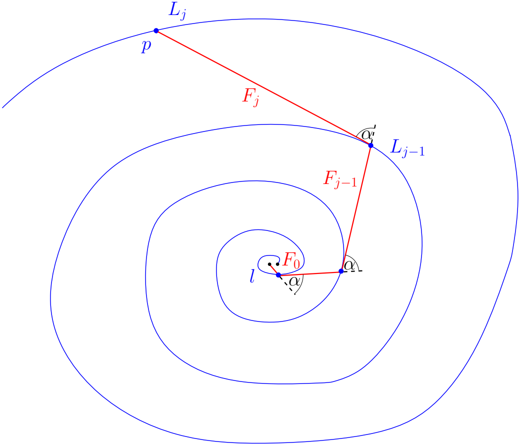
$$l_2 = \frac{A}{\cos\alpha}(e^{2\pi \cot\alpha} - 1)e^{\alpha \cot\alpha}$$

$$F_{j+1}(l_1) = \phi_{j+1}(l_1)$$
$$F_{j+1}(0) = \phi_j(l_2)$$

Q20 Explain the idea and sketch the proof!

Parameterized by lenght l of starting spirals!

$L_j(l)$ length of the curve! $F_j(l)$ (and $\phi_j(l)$) length of the free string!



Lemma 60:

$$L_{j-1} + F_j = \cos\alpha \, L_j$$

Lemma 61:

$$\frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}}$$

Upper bound by FollowFire

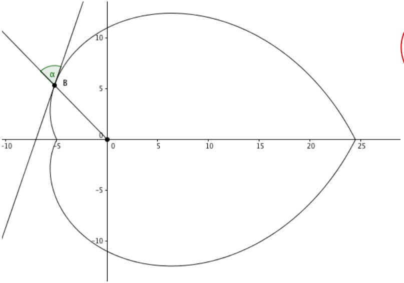
Theorem 59: FollowFire strategy is successful if $v > v_c \approx 2.6144$
Q21 Explain the meaning of these steps!

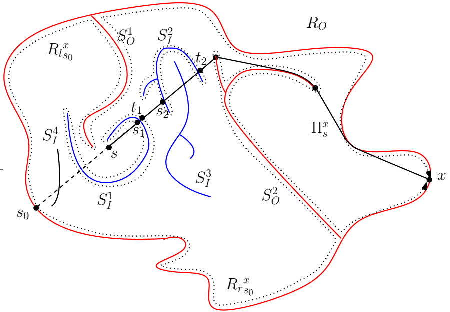
When gets the free string to zero?

1. Parameterize free strings for coil j (Linkage)
2. Structural properties
3. Successive interacting differential equations
4. Inserting end of parameter interval
5. Coefficients of power series
6. Ph. Flajolet: Singularities
7. Pringsheim's Theorem and Cauchy's Residue Theorem

General lower bounds

Theorem 68: For $v > 2$ there successful general strategy.
For $v \leq 1$ there is no such general strategy.
Q22 Present the proofs!

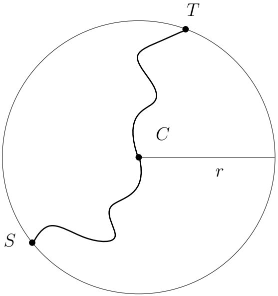


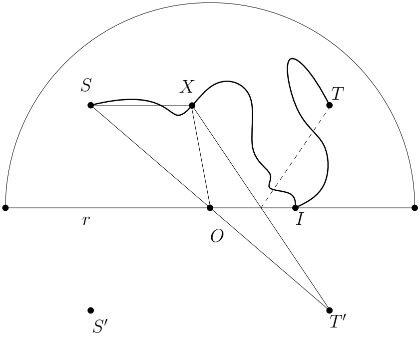


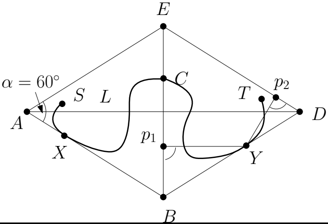
Escape path

Q23 Give the precise definition

Q24 Explain the proof for Theorem 69-71

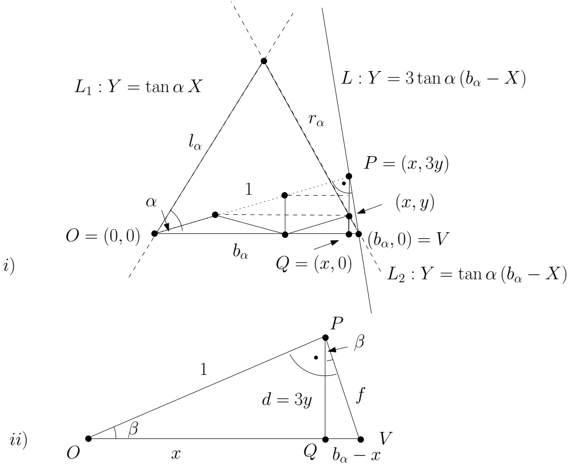




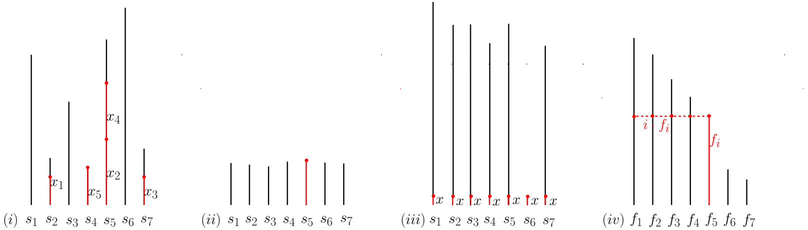


Theorem 72: There are examples where a Zig-Zag path is better than the diameter!

Q25 Sketch the construction, give the precise result!



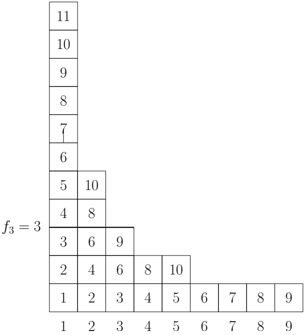
Q26 Present the idea and the definition! Proof Theorem 73!

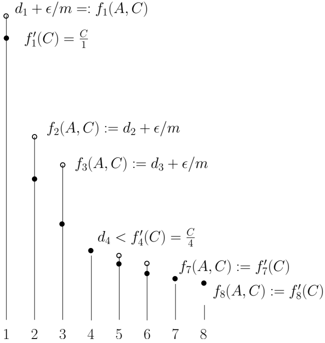


Theorem 73: For a set of sorted distances F_m (i.e. $f_1 \geq f_2 \geq \cdots \geq f_m$) we have $\text{maxTrav}(F_m) := \min_i i \cdot f_i$.

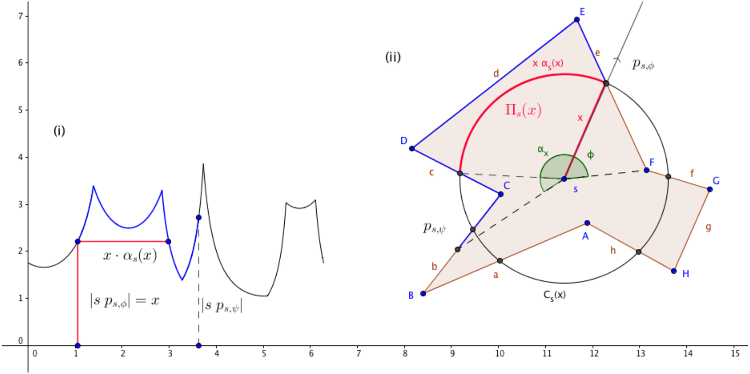
Alternative cost measure: List searching!

Theorem 74/75: The hyperbolic traversal algorithm solves problem for any list F_m with maximum traversal cost bounded by $D \cdot (\text{maxTrav}(F_m) \ln(\min(m, \text{maxTrav}(F_m)))$ for some constant D. There is a lower bound of $d \cdot C \ln \min(C, m)$ with $\text{maxTrav}(F_m(C, A)) \leq C$ for some constant d and arbitrarily large values C. Q27 Proof idea!



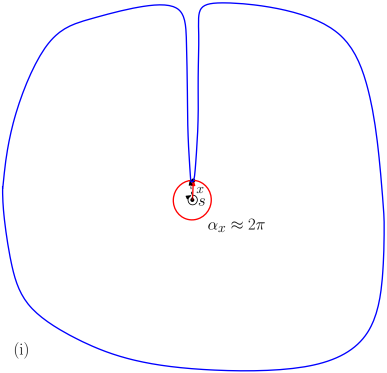


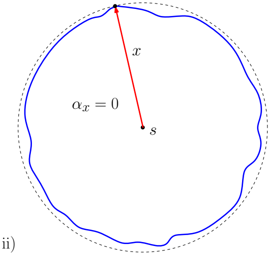
Q28 Present the idea and the definition for polygons!



Alternative cost measure: Certificate path!

Q29 Explain the extreme cases!





Alternative cost measure: Certificate path!

Sketch the proof for online approximation!

Theorem 76: There is a spiral strategy for any unknown starting point s in any unknown environment P that approximates the certificate for s and P within a ratio of 3.31864.

$$f(\gamma) = \frac{\frac{a}{\cos\beta} \cdot e^{\phi \cot\beta}}{a \cdot e^{(\phi-\gamma)\cot\beta}(1+\gamma)} = \frac{e^{\gamma\cot\beta}}{\cos\beta(1+\gamma)} \text{ for } \gamma \in [0, 2\pi] \quad (6)$$