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Probabilistic Analysis
of Algorithms
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Problem Set 6

Problem 1

A *fully polynomial-time approximation scheme (FPTAS)* for an optimization problem Π is an algorithm that gets as input an instance I of Π and some $\varepsilon > 0$. It then computes a solution of I that is within a factor $(1+\varepsilon)$ of being optimal in time polynomial in the encoding length of I and $1/\varepsilon$. Prove that the existence of an FPTAS for a strongly NP-hard linear binary optimization problem implies P=NP.

Problem 2

In the lecture we studied the smoothed complexity of linear binary optimization problems with perturbed objective functions. Let us now consider the case that one of the constraints is linear and perturbed. We consider problem instances of the following form:

$$\begin{aligned} \text{maximize} \quad & p(x) \\ \text{subject to} \quad & w^{\mathsf{T}}x = w_1x_1 + \cdots + w_nx_n \leq t, \\ & \text{and } x = (x_1, \ldots, x_n)^{\mathsf{T}} \in \mathcal{S} \subseteq \{0,1\}^n. \end{aligned}$$

Here $p : \mathcal{S} \to \mathbb{R}$ denotes an arbitrary objective function and $\mathcal{S} \subseteq \{0,1\}^n$ denotes an arbitrary set of solutions. We assume the following properties: p is injective, and if $p(x) < p(y)$ for two solutions $x, y \in \mathcal{S}$ then there exists an index $i \in [n]$ with $x_i = 0$ and $y_i = 1$.

We denote by $x^\star \in \mathcal{S}$ the optimal solution of the given instance and we define set $\mathcal{L}$ of *losers* as $\mathcal{L} = \{x \in \mathcal{S} \mid p(x) > p(x^\star)\}$, i.e., a solution is a loser if it is better than the optimal solution but cut off by the linear constraint. We assume that $\mathcal{L} \neq \emptyset$ and define the *loser gap* $\Lambda = \min_{x\in\mathcal{L}} w^{\mathsf{T}}x - t$.

Let $w_1, \ldots, w_n$ be ϕ-perturbed numbers from $[0,1]$. Prove that, for every $\varepsilon \geq 0$,

$$\mathbf{Pr}\left[\Lambda \leq \varepsilon\right] \leq n\phi\varepsilon.$$

Problem 3

Assume that a linear binary optimization problem Π with instances of the form as in Problem 2 is given. Furthermore assume that Π can be solved in pseudo-polynomial time with respect to the input length and the largest coefficient $W = \max_i w_i$. Give an algorithm with polynomial running time in the input length and ϕ that computes for any instances of Π with ϕ-perturbed coefficients w_i from $[0,1]$ an optimal solution with probability at least $1 - 1/n$.