

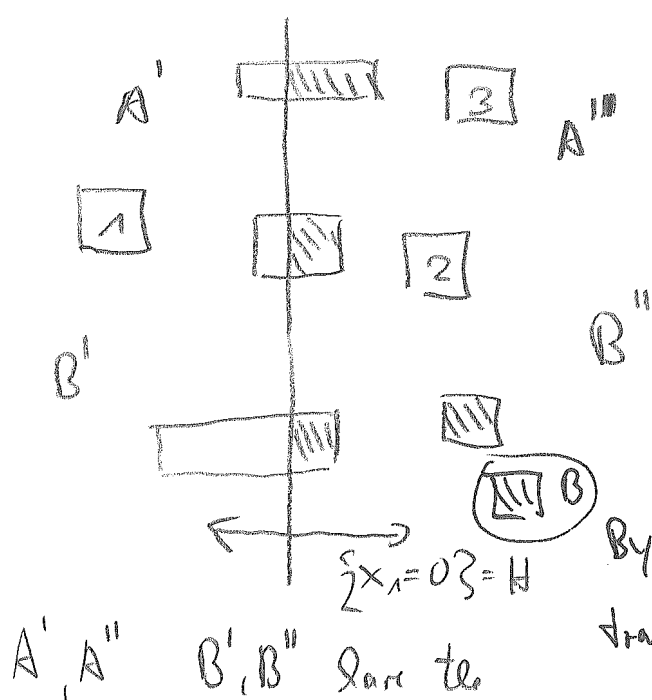
$d \geq 2$ w.o.g. $|A| \geq 2$

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Let H be a type plane in $\mathbb{R}^d$

that subdivides at least two bricks of A

w.o.g. $H = \{x_1 = 0\}$



Translate

B to B', B''

So that:

$$\text{vol}(A') = s \text{ vol}(A)$$

$$\text{vol}(A'') = (1-s) \text{ vol}(A)$$

$$\text{vol}(B') = s \text{ vol}(B)$$

$$\text{vol}(B'') = (1-s) \text{ vol}(B)$$

By translation:

Some volume relationship

[Translate does pos. of B but not $\text{vol}(A+B), \text{vol}(B)$]

$$\begin{aligned} |\text{bricks}(A')| + |\text{bricks}(B')| &< \infty \Rightarrow \text{vol}(A')^{\frac{1}{d}} + \text{vol}(B')^{\frac{1}{d}} \leq \text{vol}(A+B)^{\frac{1}{d}} \\ |\text{bricks}(A'')| + |\text{bricks}(B'')| &< \infty \Rightarrow \text{vol}(A'')^{\frac{1}{d}} + \text{vol}(B'')^{\frac{1}{d}} \leq \text{vol}(A''+B'')^{\frac{1}{d}} \end{aligned}$$

$$\text{vol}(A+B) = \text{vol}((A' \cup A'') + (B' \cup B''))$$

$$\geq \text{vol}(A'+B') + \text{vol}(A''+B'')$$

$A' \cup B'$ in $\{x_1 \leq 0\}$
 $A'' \cup B''$ in $\{x_1 \geq 0\}$
 (contains $A'+B'$ and $A''+B''$ but also other sums, $A'+B', A''+B''$ do not overlap)

(^d) and Ind. hyperk.

$$\begin{aligned}
 &> \left(\text{vol}(A')^{\frac{1}{d}} + \text{vol}(B')^{\frac{1}{d}} \right)^d + \left(\text{vol}(A'')^{\frac{1}{d}} + \text{vol}(B'')^{\frac{1}{d}} \right)^d \\
 &= \left(s^{\frac{1}{d}} \text{vol}(A)^{\frac{1}{d}} + s^{\frac{1}{d}} \text{vol}(B)^{\frac{1}{d}} \right)^d + \left((1-s)^{\frac{1}{d}} \text{vol}(A)^{\frac{1}{d}} + (1-s)^{\frac{1}{d}} \text{vol}(B)^{\frac{1}{d}} \right)^d \\
 &= s \left(\text{vol}(A)^{\frac{1}{d}} + \text{vol}(B)^{\frac{1}{d}} \right)^d + (1-s) \left(\text{vol}(A)^{\frac{1}{d}} + \text{vol}(B)^{\frac{1}{d}} \right)^d \\
 &= \left(\text{vol}(A)^{\frac{1}{d}} + \text{vol}(B)^{\frac{1}{d}} \right)^d.
 \end{aligned}$$

□

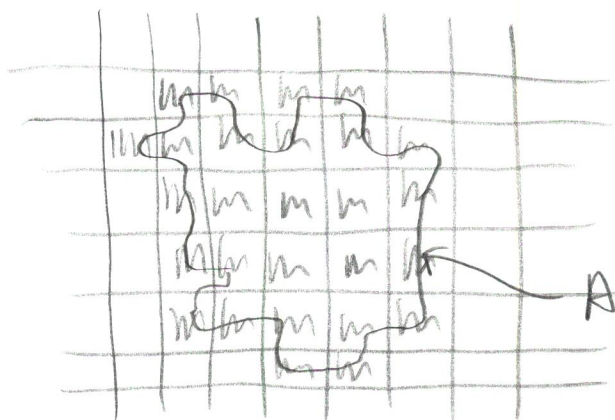
Statement holds for all brick sets!

Let A, B compact sets, Lebesgue-measurable set

(Measure theory $\rightarrow$ Standard way of assigning volume to sets in $\mathbb{R}^d$)
Volume of boxes

Consider set A (analogously for B)

Subdivision of $\mathbb{R}^d$ into "boxes" of side-length $\frac{1}{2^k}$ (hypercubes)



A_k
Set of hypercubes of side length $\frac{1}{2^k}$ intersecting with A

Approximation!

Measure theory: $\bigcap_{i=1}^{\infty} A_i =: \bigcap_k A_k = A$

$\Rightarrow \lim_{k \rightarrow \infty} \text{vol}(A_k) = \text{vol}(A)$

(also for B, B_k)

Lemma 35

$$\bigcap_k (A_k \oplus B_k) = A \oplus B$$

Proof: " $\supseteq$ " trivial any $A_k \oplus B_k$ is superset of $A \oplus B$

" $\subseteq$ " $x \in A_k \oplus B_k$ for all k

$\Rightarrow$ there are series $\{a_k\}$ and $\{b_k\}$
 $\in A_k$ $\in B_k$

$$\text{with } x = a_k + b_k \quad \forall k$$

$\Rightarrow \{a_k\} \{b_k\}$ have sub series $\{\overline{a_k}\} \{\overline{b_k}\}$
 $\uparrow$
 Bolzano-Weierstrass Def compact (bounded set)

$$\{\overline{a_k}\} \rightarrow a \quad \{\overline{b_k}\} \rightarrow b$$

A, B closed (compact) $a \in A, b \in B$

$$\Rightarrow x = a + b \in A \oplus B$$

Now Theorem 33: For general A, B

$$\text{vol}(A \oplus B)^{\frac{1}{d}} \geq \lim_{k \rightarrow \infty} \text{vol}(A_k \oplus B_k)^{\frac{1}{d}} \geq \lim_{k \rightarrow \infty} (\text{vol}(A_k)^{\frac{1}{d}} + \text{vol}(B_k)^{\frac{1}{d}})$$

Lemma 31, Measure Theory

$$A \oplus B = \bigcap_{k=0}^{\infty} (A_k \oplus B_k)$$

Bridgman's lemma (2)

$$= \text{vol}(A)^{\frac{1}{d}} + \text{vol}(B)^{\frac{1}{d}}$$

indep. / Measure Theory

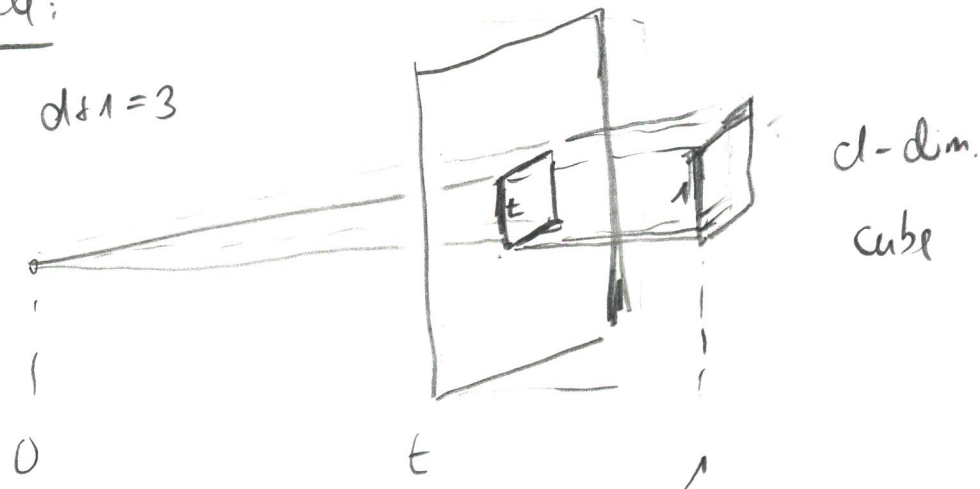
□

Observation: $d \geq 3$

$V(t)^{\frac{1}{d}}$ always concave

$v(t)$ not in general.

Example:



$$V(t) = C \cdot t^d \quad \text{convex} \quad V(t)^{\frac{1}{d}} \quad \text{concave}$$

$$= C^{\frac{1}{d}} \cdot t \quad (\text{linear})$$

linear!

Korollar 36: Let $A, B \subseteq \mathbb{R}^d$ compact sets.

$$\sqrt{\text{vol}(A) \cdot \text{vol}(B)} \leq \text{vol}\left(\frac{A \oplus B}{2}\right)$$

Proof:

$$\text{vol}\left(\frac{A \oplus B}{2}\right)^{\frac{1}{d}} \geq \text{vol}\left(\frac{A}{2}\right)^{\frac{1}{d}} + \text{vol}\left(\frac{B}{2}\right)^{\frac{1}{d}}$$

Brunn/Mink.

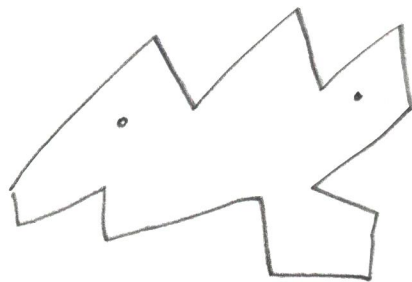
$$= \frac{1}{2} \left(\text{vol}(A)^{\frac{1}{d}} + \text{vol}(B)^{\frac{1}{d}} \right)$$

$$\geq \left(\text{vol}(A) \cdot \text{vol}(B) \right)^{\frac{1}{2d}} \quad \text{Hölder's}$$

Arith. $\geq$ geom.

□

Art Gallery + VC - dimension

Problem: Art Gallery (simple area)

polygon
12 vertices

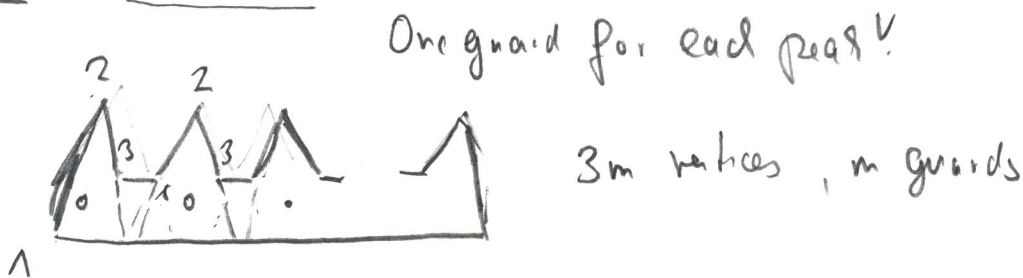
Q: Minimum number of stationary guards that "sees" full polygon?

MP-Lad Aggarwal, Leo Lin 83/86

Theorem 36 (Chvátal) $\lfloor \frac{n}{3} \rfloor$ guards are sufficient

Sometimes $\lfloor \frac{n}{3} \rfloor$ guards are necessary

Proof: Second part:

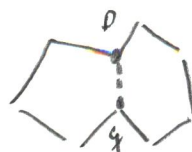


First part: Triangulation of polygon

$n-2$ triangles

$n-3$ diagonals

Induction,
existence of
diagonals



Line segment pq that cuts polygon

Triangulation: Maximum number of diagonals

Lemma 37 In any triangulation of a simple polygon P there is at least one triangle whose boundary is at most one diagonal.

$n-2$ triangles, outer boundary edges belong to unique triangles.

For $n > 3$ any triangle has at most 2 boundary edges. Disjoint boundary edges.
(n edges)

If there is one with 2 boundary edges $\Rightarrow$ claim!

If there is none with 2 boundary edges $\Rightarrow$ at most $n-2$ edges $\hookrightarrow n$ edges of P (2 such triangles)

Colorability: with three colors, place colors at the vertices

Graph structure: The vertices of an edge do not have the same colors, min. number of colors

Triangulation of P : 3 colors are enough

Does not hold for general  coloring $\leadsto$ Take the color with the smallest number of vertices

for general

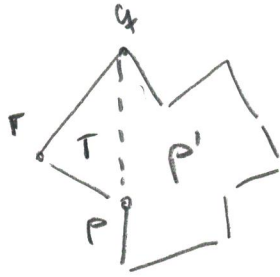
triangulations, degree!

$$\# \text{ Guards} \leq \left\lfloor \frac{n}{3} \right\rfloor \text{ (integer)}$$

Remains to show: 3 colors are enough for a triangulation

Induction: $n=3$ $\checkmark$

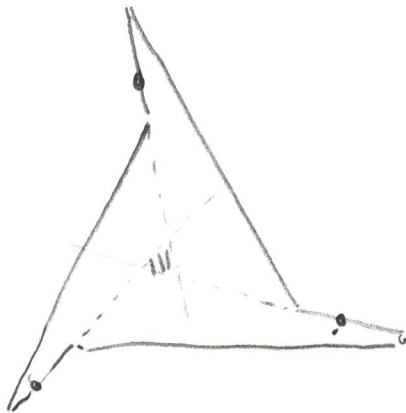
$n > 3$ Choose triangle with one diagonal.

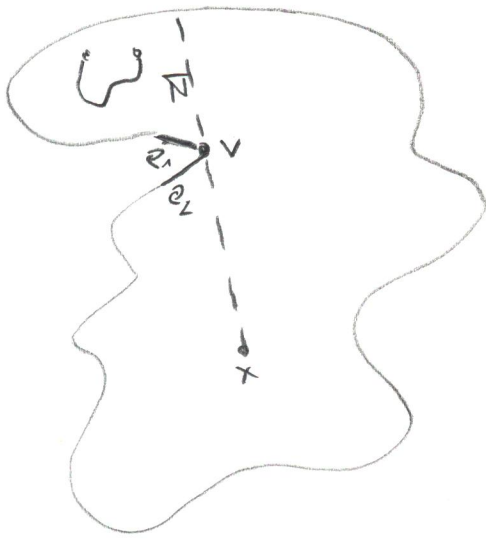
Split:

Use ind. hyp.

for P' 3 colors P and g have
different colors. r gets the third color.Ind. step ✓

□

Some simple observations P simple polygon stationary guards sees ∂P $\Rightarrow$ guards do not see the whole polygonBut: A mobile guard along path Π sees the boundary $\Rightarrow$ the guard sees the whole polygonProof:Let Π be the guarding path



Assume that

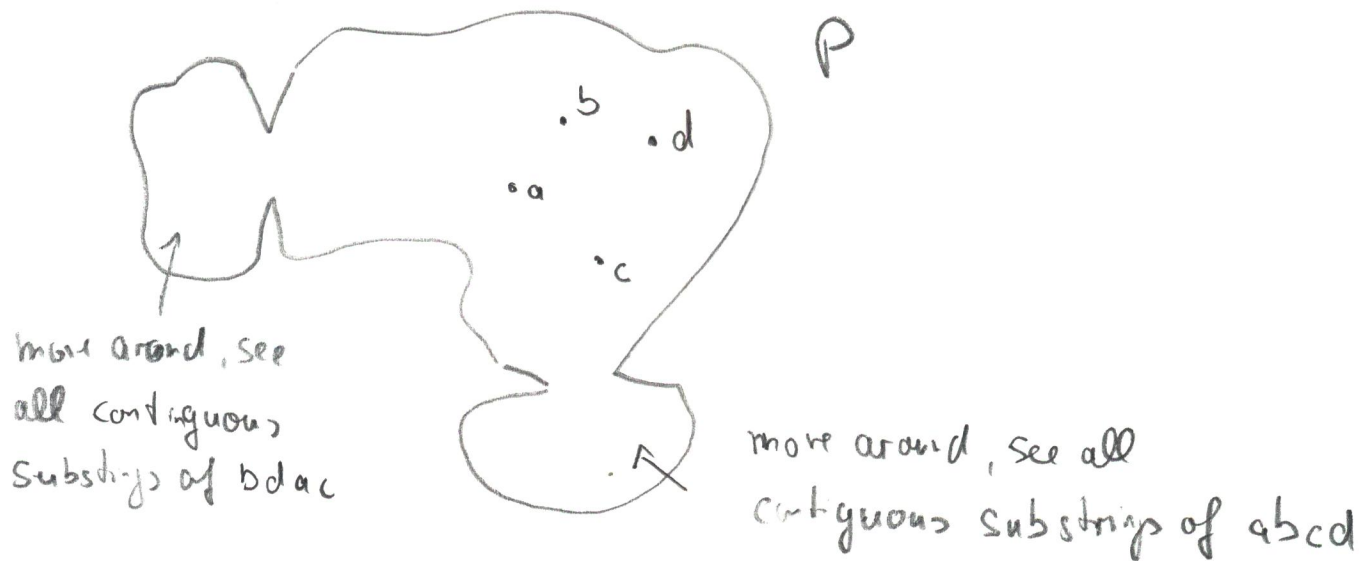
π does not "see" x

$$\Rightarrow \pi \cap \text{vis}_P(x) = \emptyset$$

$\Rightarrow \pi$ "behind" vertex v
does not see e_2

□

Walking around in a polygon with n points:
From each position some points can be seen, some not.



There are positions in P so that any subset of $\{a, b, c, d\}$
can be precisely seen.

Only b, c, a, d only bd, ac, ab, cd and so on...

Q: What is the maximum number d of points
for which such examples can be constructed

Matousek $5 \leq d < \infty$

Valtva $6 \leq d \leq 23$

Gilbert, Klein 2011 $d \leq 14$

Positive in P

from which point a

can be seen is

$\text{vis}_P(a)$

Theorem 38 (Gilbert, Klein)

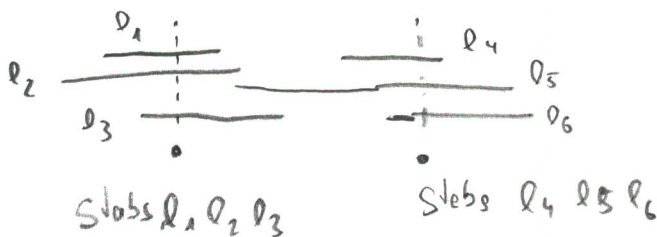
Analogous question!

$\forall P \forall 15$ points $a_1, a_2, \dots, a_{15} \in P$

not all subsets of $\{\text{vis}(a_i) \mid 1 \leq i \leq 15\}$

can be stabbed.

Stabbing: Example line segments / interval,



Intervals no more than 2



Sketch of the proof: Idea replace "ugly" vispolygons
by simple wedges.

Nice results for wedges.

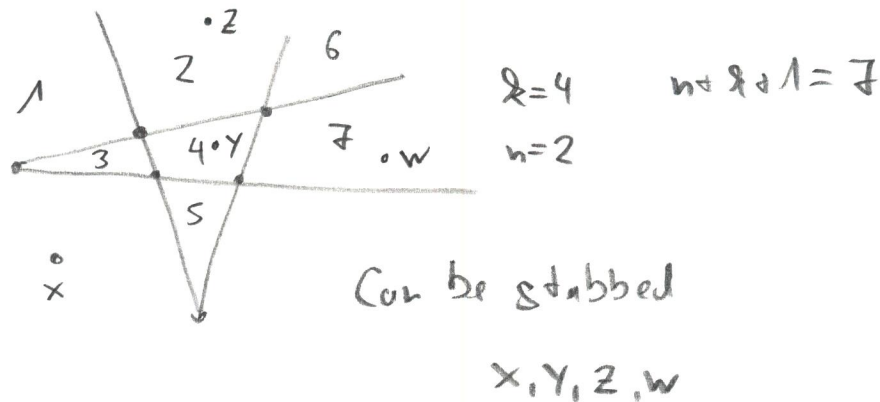
Lemma 34 (Isler et al.)

In an arrangement of ≥ 6 wedges not every subset can be stabbed.

Proof: Euler Formula $(V - E + F = C + 1)$

Arrangement of n wedges has $n + x + 1$ cells (Exercise) !
 where $x = \#$ of self or intersections

Ex.



$$n=6 \Rightarrow \max 4 \times \binom{6}{2} = 4 \cdot \binom{6}{2} = 4 \cdot \frac{6!}{2!4!} = 4 \cdot \frac{6 \cdot 5}{2} = 60 \text{ intersections}$$

$$\Rightarrow \max 6 + 60 + 7 = 67 \text{ cells}$$

Enough cells for the stabbing of $2^6 = 64$ possibilities

But some have the same subsets $\Rightarrow$ not enough cells for all subsets

Some subsets counting

(i) every wedge properly crossed by all other wedges



First and last cell contained only in W (remove one cell)

(ii) Remaining n' wedges not properly crossed
by the other wedges (but crossed by all)

Number of cells saved?

$\# |W \cap W'| = 1$ 3 intersections missing
 $\# |W \cap W'| = 2$ 2 intersections missing
 $\# |W \cap W'| = 3$

} For each
 W, W' at least
 one cell less



1 intersection
 missing +
 1 cell redundant

} For each
 W, W' at
 least one
 cell less

$\Rightarrow n'$ cells redundant or less!

all together $67 - 6 < 64$ possibilities

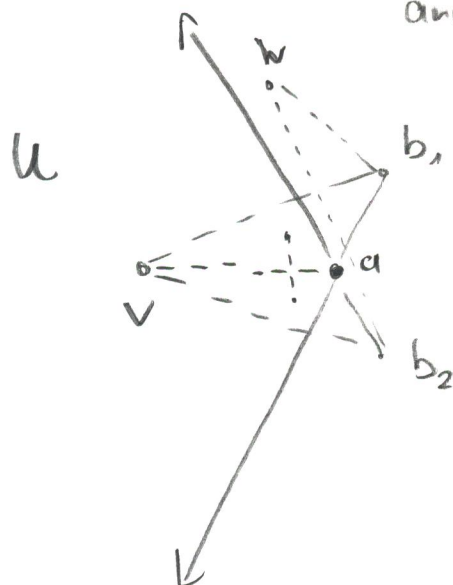
□

Proof of Theorem 38 Substitute polygons by wedges $\overline{V_i}$

Suppose $\exists P, S \subset P$ $|S| = 15$ and S is "shattered"

$\Rightarrow \forall \overline{V} \subseteq S \exists v_T \in P: \overline{V} \cap S = \overline{V}$

Let $a, b_1, b_2 \in S$ $v = v_T$ where $a \in T$, $b_1, b_2 \in \bar{T}$
and $a \in \text{fria}(v, b_1, b_2)$

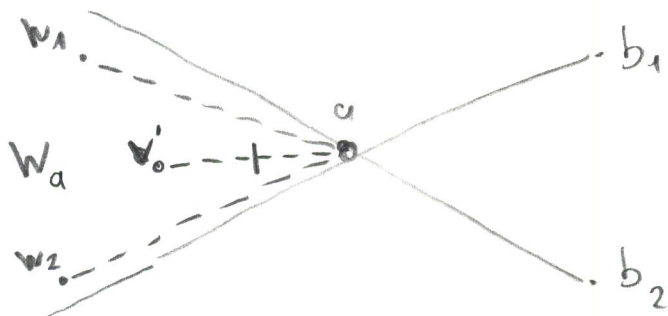


Claim I: w sees b_1, b_2
 $\Rightarrow w \in U$ (wedge)

Proof: Otherwise $\overline{va}$
"surrounded", not a
simple polyge

Let w_1, w_2 be the outermost points of $V := \{v_T \mid \Delta \in S\}$
that sees a, b_1, b_2 ($\exists$ by assumpt. $\in U$ by claim I)

Let W denote the wedge formed by a, w_1, w_2 .



Def: $V_{\{b_1, b_2\}} := \{v_T \in V \mid \{b_1, b_2\} \in T\}$

Claim II: $V_{\{b_1, b_2\}} \cap W_a = V_{\{b_1, b_2\}} \cap \text{vis}(a)$

" $\supseteq$ " Claim I, Def. of W_a

(94)

" $\subseteq$ " Let $v' \in V_{\{b_1, b_2\}} \cap W_a$

Assume $\Rightarrow \overline{v'}$ does not see a

$\Rightarrow \overline{v'a}$ is surrounded $\Downarrow$

$\Rightarrow v'$ sees a $v' \in \text{vis}(a)$ $\square$

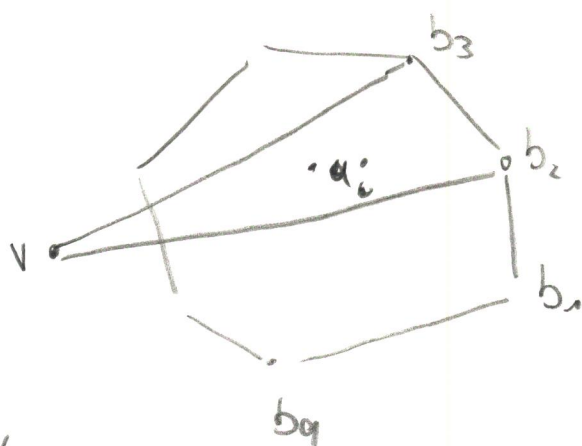
Proof of Theorem consists of 2 parts

$\boxed{\text{I}}$ ≤ 5 points inside $ch(s)$

$\boxed{\text{II}}$ ≤ 9 points on $ch(s)$

Proof only for $\boxed{\text{I}}$

Assume $a_1, \dots, a_6$ inside $ch(s)$, $b_1, \dots, b_6$ on $ch(s)$



Let $v := v_{\{b_1, \dots, b_6\}}$

Situation of Claim $\square \Rightarrow \exists$ wedy w_i
for $i=1, \dots, 6$

$$V_{\{b_2, b_3\}} \cap w_i = \overline{V_{\{b_2, b_3\}}} \cap \text{vis}(a_i)$$

$\Rightarrow$

$$\forall i : V_{\{b_1, \dots, b_q\}} \cap W_i = V_{\{b_1, \dots, b_q\}} \cap \text{vis}(a_i)$$

By assumption: $\forall A \subseteq \{a_1, \dots, a_6\} \exists V_{B \cup A} \in V$

$$B := \{b_1, \dots, b_q\}$$

$\Rightarrow V_{B \cup A}$ stabs exactly to vispolygons of A

$\Rightarrow V_{B \cup A}$ stabs exactly to edges of A $\downarrow$
Lemma 39

Proof of $\square$ very complicated, similar argument!

□

Definition 40

Let $(X, \mathcal{F})$ be a set system of a ground set X .
 $\mathcal{F} \subseteq \mathcal{P}(X)$.

(i) $A \subseteq X$ is shattered by $\mathcal{F} : \Leftrightarrow$

$$\forall B \subseteq A \quad \exists F \in \mathcal{F} : B = F \cap A$$

(in other words $\mathcal{F}|_A := \{F \cap A \mid F \in \mathcal{F}\} = \mathcal{P}(A)$)

(ii) Vapnik-Chervonenkis dimension of $(X, \mathcal{F})$:

$$\dim_{VC}(\mathcal{F}) := \max \text{ size of a shattered set} \\ (= \infty \text{ possible})$$

Examp. (eg. 1) $X =$ all points in a simple polygon

$$\mathcal{F} = \{ \text{vis}(a) \mid a \in X \} \Rightarrow \dim_{VC}(\mathcal{F}) \leq 14 \\ \geq 6.$$

2) $X = \mathbb{R}^2$ $\mathcal{F} =$ Set of all closed half-spaces in $\mathbb{R}^2$

$$\dim_{VC}(\mathcal{F}) = 3$$

• can be shattered
 • by half spaces!

4 points

(97)



non-convex

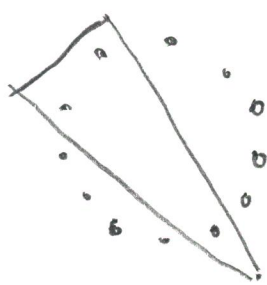


convex

not separable

positions

3) $X = \mathbb{R}^2$ $\tilde{\tau} =$ all convex sets in $\mathbb{R}^2$

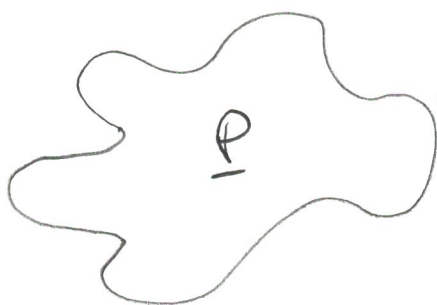


The vertex set of
any regular n -gon
can be shattered

$n \rightarrow \infty$

$\dim_{VC}(\tilde{\tau}) = \infty$

Recap: Art gallery



Jordan curve
(no holes)

Fixed number of Guards? Q_1
No $\leadsto \lfloor \frac{n}{3} \rfloor$ example



$Q_2 :=$ Assume that for any point p in P at least
 $\frac{1}{r}$ times $\text{vol}(P)$ can be seen.

Is there a bounded number (for example r) of
guards that are sufficient?

Proposition, P requires w guards

$$: \Leftrightarrow \text{set system } \left(\{ \text{vis}_P(p) \mid p \in P \} = \mathcal{F} \right)$$

$(P, \mathcal{F})$ can be stabbed with w points

Definition 4.1 $\mathcal{F}$ set system over X . $(X, \mathcal{F})$

(i) A subset $\nabla \subseteq X$ is called a transversal of $\mathcal{F}$ if ∇ intersect all sets of $\mathcal{F}$

$$\forall F \in \mathcal{F} \wedge F \neq \emptyset : F \cap \nabla \neq \emptyset$$

(ii) The transversal number $\tau(\mathcal{F})$ is the smallest possible cardinality of a transversal of $\mathcal{F}$.

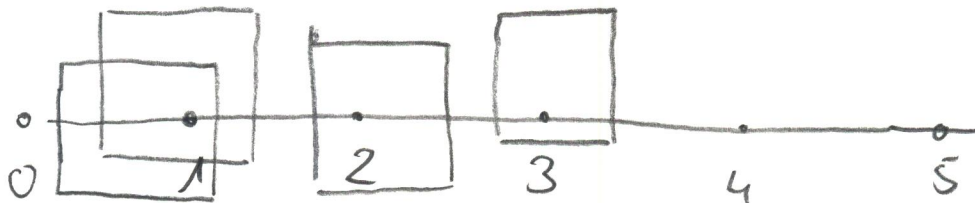
— Such transversals always exists:

Simply choose $\nabla = X$

— Our example $\tau(\{ \text{vis}_P(p) \mid p \in P \})$ smallest number of guards!

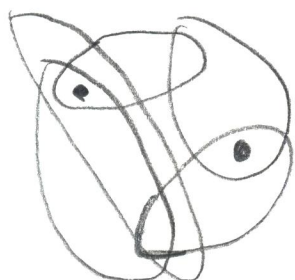
— Example $\tau(\mathcal{F}) = \infty$

$X = \mathbb{R}^2$ $\mathcal{F} = \{ \text{all closed unit squares that hit the } x\text{-axis} \}$



$\mathbb{Z}$ is a transversal

- Example:



$$\tilde{T} = 2$$

Many problem can be redefined by questions about transversals and its numbers

Another important parameter

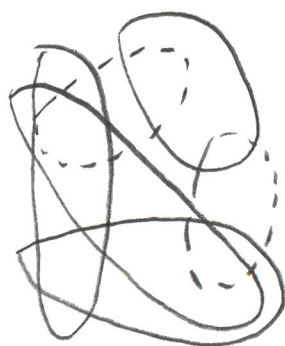
Definition 42 $\tilde{T}$ set system over X . $(X, \tilde{T})$

(i) A subset $M \subseteq \tilde{T}$ is called a packing (matching)

$$\Leftrightarrow \forall F_1, F_2 \in M \Rightarrow F_1 \cap F_2 = \emptyset$$

(ii) The packing number $\nu(\tilde{T})$ is the maximal cardinality of a packing of $\tilde{T}$.

- Ex.



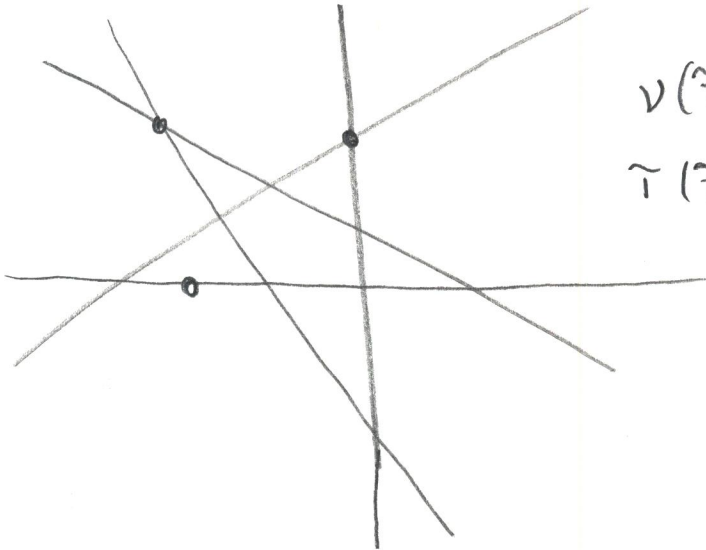
$$\nu = 2$$

Does always $v(\tilde{T}) = \tilde{\tau}(\tilde{T})$ hold?

(100)

No, example $v < \tau$

$X = \mathbb{R}^2$, $\tilde{T} = \{n \text{ lines in general position}\}$



$$v(\tilde{T}) = 1$$

$$\tilde{\tau}(\tilde{T}) = \lceil \frac{n}{2} \rceil$$