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# AN OPTIMAL COMPETITIVE STRATEGY FOR WALKING IN STREETS*

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**Abstract.** A simple polygon $P$ with two distinguished vertices, $s$ and $t$, is called a *street* if the two boundary chains from $s$ to $t$ are mutually weakly visible. We present an on-line strategy that walks from $s$ to $t$, in any unknown street, on a path at most $\sqrt{2}$ times longer than the shortest path. This matches the best lower bound previously known and settles an open problem in the area of competitive path planning. (The result was simultaneously and independently obtained by the first three authors and by the last two authors. Both papers, [C. Icking, R. Klein, and E. Langetepe, *Proceedings of the* 16*th Symposium on Theoretical Aspects in Computer Science*, Lecture Notes in Comput. Sci. 1563, Springer-Verlag, Berlin, 1999, pp. 110–120] and [S. Schuierer and I. Semrau, *Proceedings of the* 16*th Symposium on Theoretical Aspects of Computer Science*, pp. 121–131], were presented at STACS'99. The present paper contains a joint full version.)

**Key words.** computational geometry, autonomous robot, competitive strategy, LR-visibility, on-line navigation, path planning, polygon, street

**AMS subject classifications.** 68Q17, 68Q25, 68R01, 68W40

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## 1. Introduction.

The path planning problem of autonomous mobile robots has received a lot of attention in the communities of robotics, computational geometry, and on-line algorithms; see, e.g., Rao et al. [37], Blum, Raghavan, and Schieber [6], and the surveys by Berman [4] in Fiat and Woeginger [15] and by Mitchell [35] in Sack and Urrutia [38].

In on-line navigation one has to perform a certain task in an initially unknown environment. We are interested in strategies that are correct, in that the objective will always be achieved whenever this is possible, and in performance guarantees of the following kind. Given a navigation problem $Q$, we want to relate the cost of solving any problem instance $P \in Q$ by means of strategy $S$ to the cost of solving $P$ optimally, using an off-line strategy. If the former never exceeds the latter times a certain constant factor, $c$, then strategy $S$ is said to be a *$c$-competitive* solution of $Q$; this notion was coined by Sleator and Tarjan in their seminal paper [43]. Surveys on general on-line algorithms can be found in Fiat and Woeginger [15] and Ottmann, Schuierer, and Hipke [36].

Given an on-line problem $Q$, three questions arise: Does a competitive solution exist? If $S$ is a solution, what is its true *competitive factor*, i.e., the smallest $c$ such that $S$ is $c$-competitive? And finally, what is the smallest factor $c$ that can be attained by *any* strategy solving $Q$? This number is called the *competitive complexity* of problem $Q$.

There are not so many navigation tasks we are aware of whose competitive com-

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plexities are precisely known. One of them is searching for a point on $m$ halflines that meet at the start point; see Baeza-Yates, Culberson, and Rawling [2] and also Bellman [3], Gal [16], Schuierer [39], López-Ortiz and Schuierer [34], and Alpern and Gal [1].

Another one is looking around a corner in a wall; see Icking, Klein, and Ma [22]. Recently, matching lower and upper bounds for some restricted on-line TSP problems where shown by Blom et al. [5]. More often, there is a gap between the smallest competitive factor known and the best lower bound, such as in the polygon exploration problem; see Hoffmann et al. [18].

In this paper we prove that walking in an unknown street is of competitive complexity $\sqrt{2}$, thus settling a problem that has been open for a decade. A *street* is a simple polygon $P$ with two vertices, $s$ and $t$, that mark the start and target point of the walk, subject to the condition that the two boundary chains connecting $s$ to $t$ are mutually weakly visible[1]; see Figure 1.1 for an example. This is equivalent to saying that from each $s$-to-$t$ path inside $P$, each point of $P$ can be seen at least once. Streets were introduced in Klein [24] to model racetracks and rivers like the Rhine that may contain curves and bays but no cul-de-sacs winding away from the main route. It was shown in [24, 25] that there exists a strategy that is competitive with a factor of 5.72, and that no factor smaller than $\sqrt{2}$ can be achieved, not even by a randomized strategy.

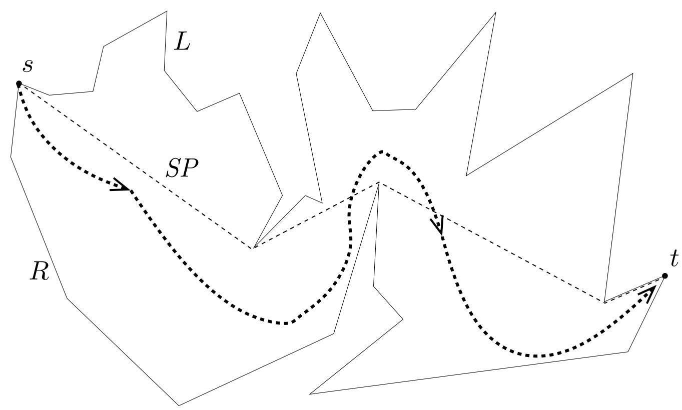


FIG. 1.1. *A street.*

Since then, the street problem has attracted considerable attention. Some research was devoted to structural properties. Tseng, Heffernan, and Lee [44] have shown how to report all pairs of vertices $(s,t)$ of a given polygon for which it is a street; for star-shaped polygons many such vertex pairs exist. Das, Heffernan, and Narasimhan [10] have improved on this result by giving an optimal linear time algorithm. Ghosh and Saluja [17] have described how to walk an unknown street with a minimum number of turns.

For arbitrary polygons it is quite easy to see that in general no strategy can guarantee a search path whose length is at most a constant times the length of the shortest path from start to target.[2] Therefore, some researchers have designed competitive search strategies for classes of polygons more general than streets; see Datta

[1]Two sets are mutually weakly visible if each point of one set can see at least one point of the other.

[2]If $n$ L-shaped legs of unit length lead away from a central place, the search path can be of length $2n-1$, in the worst case, while the shortest path is of length 1.

and Icking [13, 14], Datta, Hipke, and Schuierer [12], and López-Ortiz and Schuierer [30, 33].

Other authors have shown that more general search problems can be solved for the original street polygons. Indeed, not only can we walk to vertex $t$ starting from vertex $s$, where $s, t$ are the two special vertices defining the street, but it has also been shown in Bröcker and Schuierer [8] and Bröcker and López-Ortiz [7] that one can find any boundary point from any starting point on the boundary of a street by means of a 69.216-competitive strategy.

Carlsson and Nilsson [9] have shown that the art gallery problem remains NP-hard for street polygons. However, the problem of computing the minimum number of guards located on a given watchman route can be solved very efficiently for streets, while it is NP-hard for general polygons.

Other research has addressed the gap between the $\sqrt{2}$ lower bound and the first upper bound of 5.72 for the original street problem. The upper bound was lowered to 4.44 in Icking [19], then to 2.61 in Kleinberg [26], to 2.05 in López-Ortiz and Schuierer [29], and to 1.73 in López-Ortiz and Schuierer [31]. López-Ortiz and Schuierer [32] showed that a particular strategy called *CAB* has the true competitive ratio 1.6837. Using different search strategies, the upper bound was further decreased to 1.57 in Semrau [42], and to 1.51 in Icking et al. [23]. Further attempts were made by Dasgupta, Chakrabarti, and DeSarkar [11] and by Kranakis and Spatharis [27].

But it remained an open question whether there existed a search strategy with optimal competitive factor $\sqrt{2}$; this was mentioned open problem no. 13 in Mitchell [35].

In this paper the problem is finally solved. We introduce a new strategy and prove that the search path it generates, in any particular street, is at most $\sqrt{2}$ times the length of the shortest path from $s$ to $t$. Unlike many approaches discussed in previous work, the optimal strategy we are presenting here is not a mere artifact. Rather, its definition is well motivated by backward reasoning, as we shall now explain.

The crucial subproblem can be parametrized by a single angle, $\phi$. For each possible value of $\phi$ a lower bound can be established; see section 3.1. For the maximum value $\phi = \pi$ the existence of a strategy matching this bound follows from the properties of a street. We state a requirement in section 3.2 that would allow us to extend an optimal strategy from a given value of $\phi$ to smaller values. From this requirement we can infer how the strategy should proceed; see section 3.3. One of the problems is to prove that the requirement can be fulfilled; see section 3.4.

**2. Definitions and basic properties.** First, we briefly recall some basic definitions.

A simple polygon $P$ is considered as a room, with its edges as opaque walls. By $\partial P$ we denote the boundary of polygon $P$. Two points inside $P$ are mutually *visible*, i.e., see each other, if the connecting line segment is contained within $P$. As usual, two sets of points are said to be *mutually weakly visible* if each point of one set can see at least one point of the other set.

DEFINITION 2.1. *A simple polygon $P$ in the plane with two distinguished vertices $s$ and $t$ is called a* street *if the two boundary chains from $s$ to $t$ are weakly mutually visible; for an example, see Figure* 1.1. *Streets are sometimes also called* LR-visible polygons [10, 44]*, where $L$ and $R$ denote the left and the right boundary chains from $s$ to $t$, respectively.*

A strategy for searching a target in an unknown street is an on-line algorithm for a mobile robot, modeled by a point, that starts at vertex $s$, moves around inside the polygon, and eventually arrives at the target, $t$. The robot is equipped with a vision system that provides the visibility polygon for its actual position at each time, and everything that has been visible is memorized. When the target becomes visible the robot goes there, and its task is accomplished.

As the room's floor plan is not known in advance, the robot's path can be longer than the shortest path, $SP$, from $s$ to $t$ inside $P$. Our goal is to bound that detour.

DEFINITION 2.2. *A strategy for searching a target, $t$, in a street is called* competitive with factor $c$ *(or $c$-*competitive*, for short) if its path is never longer than $c$ times the length of the shortest path from $s$ to $t$.*

All existing algorithms for walking along a street are making use of some geometric properties that can be derived from the definition of a street; these facts and their complete proofs can be found in [25]. For the convenience of the reader these properties, together with an outline of their proofs, will now be stated.

First, we consider the situation at the beginning. As the robot starts from vertex $s$, it may not be able to see the whole polygon. The parts invisible to the robot are called *caves*. Each cave is hidden behind a reflex vertex, $v$, which is one whose internal angle exceeds 180°. Such a reflex vertex—and its associated cave—is called *left* if its adjacent edges in $\partial P$ lie to the left of the ray emanating from the robot's position through $v$. Right reflex vertices are defined analogously.

If $s$ is the start point of a street, these caves can occur only in a certain order. As the robot scans $\partial P$ in a clockwise direction, it encounters a consecutive sequence of left caves with left reflex vertices $v_l^{-a}, v_l^{-a+1}, \ldots, v_l^0, v_l^1$ followed by a consecutive sequence of right caves with right reflex vertices $v_r^1, v_r^0, \ldots, v_r^{-b+1}, v_r^{-b}$; see Figure 2.1. Either sequence can be empty. The reason for this ordering is that a right cave cannot be predecessor of a left cave. Assuming the contrary, let $v$ be a right reflex vertex that appears before the left reflex vertex $w$ in clockwise order on $\partial P$. Let $v^-$ be the predecessor vertex of $v$, and let $w^+$ denote the successor of $w$. If $v$ were a vertex of chain $L$, then $v^-$ would not be able to see a point of chain $R$, in contradiction to the street property. Thus, $v \in R$ holds. Similarly, we have $w \in L$. But this is impossible since $L$ and $R$ are connected and meet at $s$.

Of all the left caves visible from $s$ only the clockwise-most can contain the target; see Figure 2.1. The reason is similar to the proof above. In fact, the reflex vertex $v_l$ of the clockwise-most left cave cannot belong to chain $R$, or its successor on the boundary would not be able to see a point of $L$. Analogously, only the counterclockwise-most right cave with reflex vertex $v_r$ can contain the target vertex $t$.

Consequently, if only $v_l$ exists, then its cave must contain the target, and the robot walks straight to $v_l$; see Figure 2.1(ii). We observe that this reflex vertex must also be visited by the shortest path from $s$ to $t$. The same holds if only $v_r$ exists.

A more interesting situation arises if both $v_l$ and $v_r$ exist. Then the target can be situated in either of their caves, but the robot does not know in which one; see Figure 2.1(i). We call this a *funnel situation*. The angle, $\phi$, between the directions from the actual position to $v_l$ and to $v_r$ is called the *opening angle*; it is always smaller than $\pi$, as another consequence of the street property. Most search strategies cause the robot to walk into this funnel of angle $\phi$. They differ in choosing the direction into the funnel.

As the robot leaves the start vertex $s$, the vertices $v_l$ and $v_r$ are maintained by the robot. Essentially, $v_l$ is the reflex vertex of the clockwise-most left cave in front of the robot, and $v_r$ is the entrance vertex of the counterclockwise-most right cave. The vertices $v_l$ and $v_r$ are known to belong to $L$ and $R$, respectively; but the horizon,

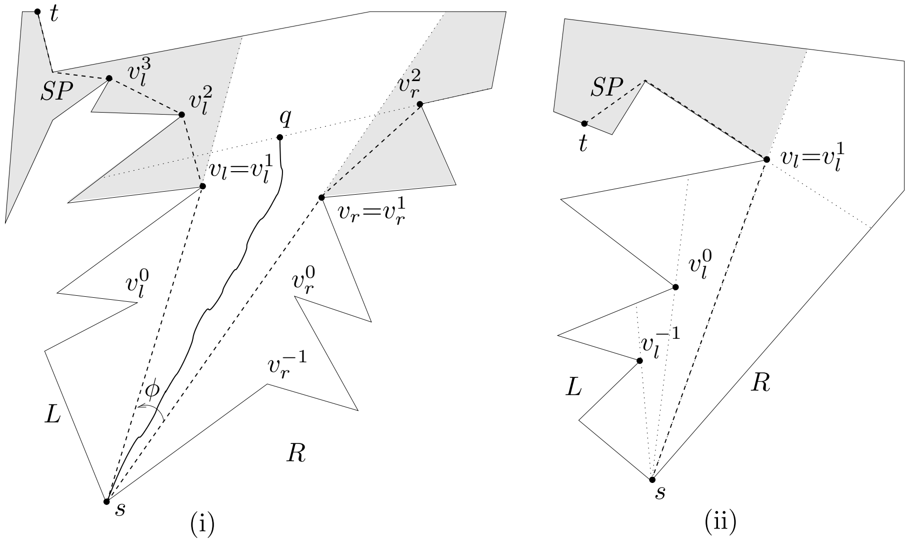


FIG. 2.1. *Typical situations in streets.*

that is, the boundary part between $v_l$ and $v_r$, can belong to either chain, depending on the position of the target.

To summarize, the robot behaves as follows. If the target is visible from the robot's current position, the robot walks straight to the target. If only one of the vertices $v_l, v_r$ exists, then the robot walks straight to this vertex, which is also visited by the shortest path. If both $v_l$ and $v_r$ exist, then the robot walks into the funnel defined by its current position, $c_0$, and by $v_l$ and $v_r$.

As the robot moves into the funnel, three events can happen. The most important event occurs when a new reflex vertex, say $v_l{}'$, appears behind vertex $v_l$. In this case, we know from the discussion above that the target cannot be contained in the cave of $v_l{}'$; it must be situated in the caves of $v_l{}'$ or $v_r$. Now the robot proceeds with $v_l{}'$ and $v_r$. This event can occur repeatedly on both sides. It generates convex chains of reflex vertices $v_l^1, v_l^2, \ldots, v_l^m$ and $v_r^1, v_r^2, \ldots, v_r^n$ that form a funnel with apex $c_0$.

Another event can occur when one of the two innermost caves, say the left, becomes completely visible. In this case, the target must be situated in the right cave, and the robot walks to its associated reflex vertex, $v_r^n$. On reaching $v_r^n$, the funnel situation is resolved, and we know that the chain $v_r^1, v_r^2, \ldots, v_r^n$ belongs to the shortest path from $s$ to $t$.

The third event occurs when the target becomes visible, e.g., in the right cave; then the robot walks straight to $t$, visiting the reflex vertex $v_r^n$ on the way.

This analysis shows that detour is only caused by funnels, and that the overall competitive factor of a search strategy for streets depends only on its performance in funnels.

As a consequence, we can restrict our attention to the subclass of funnel polygons. They consist of two chains of reflex vertices with a common start point $s$; see Figure 2.2 for an example. The two reflex chains end in vertices $t_l$ and $t_r$, respectively, and the line segment $t_l\,t_r$ closes the polygon. A funnel polygon represents a funnel situation in which the target $t$ lies arbitrarily close behind either $t_l$ or $t_r$, and the strategy will know which case applies only when the line segment $t_l t_r$ is reached. For analyzing

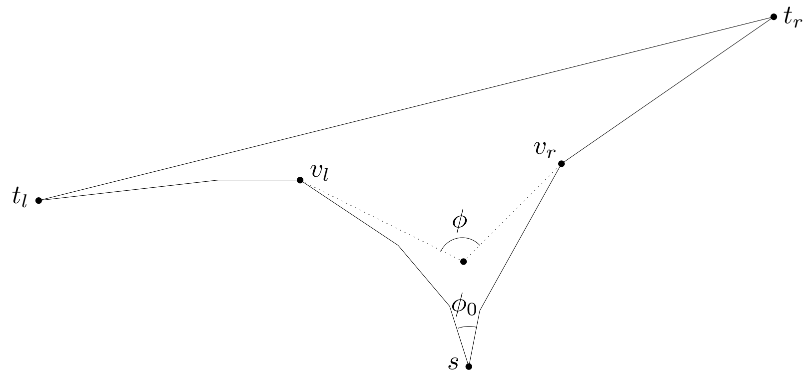


FIG. 2.2. *A funnel.*

a strategy, both cases have to be considered and the worse of them determines the competitive factor. Other funnel situations, which end before line segment $t_l t_r$ is reached or where the goal is further away from $t_l$ or $t_r$, will produce a smaller detour.

Since the walking direction is always within the opening angle, $\phi$ is always *strictly increasing*. It starts at the angle, $\phi_0$, between the two edges adjacent to $s$, and reaches, but never exceeds, $180°$ when finally the goal becomes visible. By this property, it is quite natural to take the opening angle $\phi$ for *parameterizing a strategy*.

**3. A strategy which always takes the worst case into account.** In the previous section we have seen that a crucial situation occurs when the robot is faced with two caves, one left and one right, and does not know in which of them the target, $t$, is situated. This situation can be parametrized by the funnel's opening angle $\phi$.

Let us assume that $\frac{\pi}{2} \leq \phi$ holds. We will see in section 3.1 that for each value of $\phi$, a *lower* bound for the competitive ratio is given by

$$K_\phi = \sqrt{1 + \sin \phi}.$$

If $\phi = \pi$, then $K_\pi = 1$. The street properties ensure that the robot is able to look into the caves and see the target. Hence, the optimal strategy is given by walking straight to $t$. That means, for $\phi = \pi$ we have a strategy matching the lower bound.

Now assume that $\frac{\pi}{2} \leq \phi_1 < \phi_2 < \pi$ holds, and that we have already found a strategy with (optimum) competitive factor $K_{\phi_2}$ for all opening angles $\geq \phi_2$. We would like to extend it to a $K_{\phi_1}$-competitive strategy for opening angles $\phi_1$. This is possible iff a certain geometric condition can be met, which will be stated in section 3.2. This condition gives rise to a certain curve (see section 3.3), and this curve will then be shown to have the required properties in section 3.4. Finally, in section 3.5, we deal with opening angles less than $\pi$.

**3.1. A generalized lower bound.** We start with a generalized lower bound for initial opening angles $\geq \frac{\pi}{2}$. For an arbitrary angle $\phi$, let

$$K_\phi := \sqrt{1 + \sin \phi}\,.$$

LEMMA 3.1. *Assume an initial opening angle $\phi_0 \geq \frac{\pi}{2}$. Then no strategy can guarantee a smaller competitive factor than $K_{\phi_0}$.*

*Proof.* We take an isosceles triangle with an angle $\phi_0$ at vertex $s$; the other vertices are $t_l$ and $t_r$; see Figure 3.1.

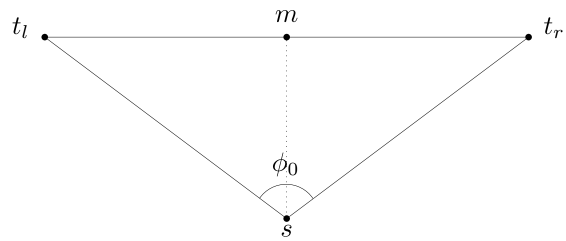


FIG. 3.1. *Establishing a generalized lower bound.*

The goal becomes visible only when the line segment $t_l\, t_r$ is reached. If this happens to the left of the midpoint $m$, the goal may be to the right, and vice versa. In any case the path length is at least the distance from $s$ to $m$ plus the distance from $m$ to $t_l$. For the ratio, $c$, of the path length to the shortest path we obtain by simple trigonometry

$$c \geq \cos\frac{\phi_0}{2} + \sin\frac{\phi_0}{2} = \sqrt{1+\sin\phi_0} = K_{\phi_0}\,. \qquad \square$$

For $\phi_0 = \frac{\pi}{2}$, we have the well-known lower bound of $\sqrt{2}$ stemming from a rectangular isosceles triangle; see Klein [25].

Note that the lower bound $K_{\phi_0}$ also applies to any nonsymmetric situation, since at the start the funnel is unknown except for the two edges adjacent to $s$, and it may turn into a nearly symmetric case immediately after the start. In other words this means that for an initial opening angle $\phi_0$, a competitive factor of $K_{\phi_0}$ is always the best we can hope for.

In the following we develop a strategy that is $K_\phi$-competitive in all funnel polygons of opening angle $\phi$.

**3.2. Sufficient requirements for an optimal strategy.** In a funnel with opening angle $\pi$ the goal is visible and there is a trivial strategy that achieves the optimal competitive factor $K_\pi = 1$. So we look backwards to decreasing angles.

Let us assume for the moment that the funnel is a triangle, and that we have a strategy with a competitive factor of $K_{\phi_2}$ for all triangular funnels with initial opening angle $\phi_2$. How can we extend this to initial opening angles $\phi_1$ with $\pi \geq \phi_2 > \phi_1 \geq \frac{\pi}{2}$?

Starting with an angle $\phi_1$ at point $p_1$ we walk a certain path of length $w$ until we reach an angle of $\phi_2$ at point $p_2$, from where we can continue with the known strategy; see Figure 3.2. We assume for the moment that the left and right reflex vertices, $v_l$ and $v_r$ as defined in section 2, do not change.

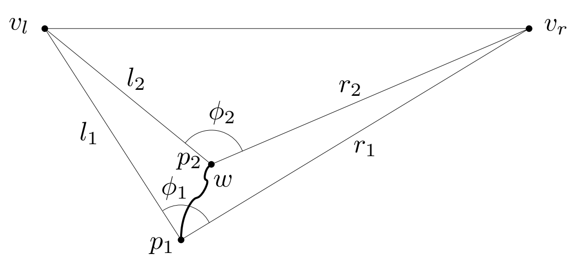


FIG. 3.2. *Getting from angle $\phi_1$ to $\phi_2$.*

Let $l_1$ and $l_2$ denote the distances from $p_1$, respectively, $p_2$, to $v_l$ at the left side, and $r_1$ and $r_2$ the corresponding distances at the right. If $t = v_l$, the length of the robot's path from $p_1$ to $t$ is not greater than $w + K_{\phi_2} l_2$. If now $K_{\phi_1} l_1 \geq w + K_{\phi_2} l_2$ holds and the analogous inequality $K_{\phi_1} r_1 \geq w + K_{\phi_2} r_2$ for the right side, we have a competitive factor not bigger than $K_{\phi_1}$ for triangles with initial opening angle $\phi_1$. By combining the two inequalities we can express the condition as

$$w \leq \min(K_{\phi_1} l_1 - K_{\phi_2} l_2, K_{\phi_1} r_1 - K_{\phi_2} r_2)\,, \tag{3.1}$$

which will prove useful later on.

Note that condition (3.1) is additive in the following sense. If it holds for a path $w_{12}$ from $\phi_1$ to $\phi_2$ and for a continuing path $w_{23}$ from $\phi_2$ to $\phi_3$, it is also true for the combined path $w_{12} + w_{23}$ from $\phi_1$ to $\phi_3$. This will turn out to be very useful: if (3.1) holds for arbitrarily small, successive steps $w$, then it is also true for all bigger ones.

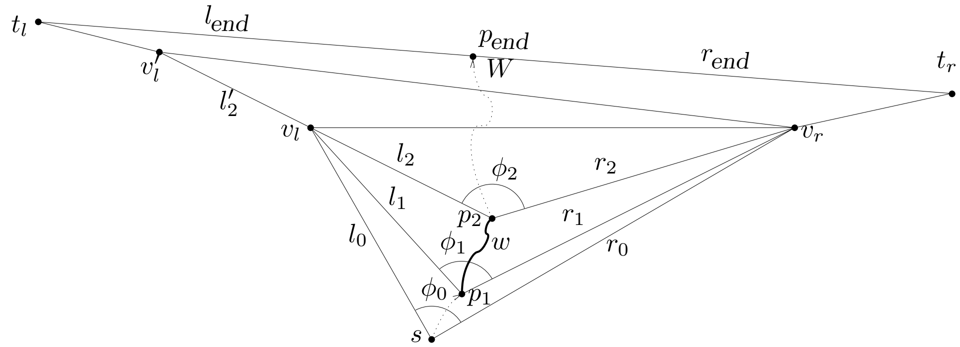


FIG. 3.3. *When $p_2$ is reached, the most advanced visible point to the left jumps from $v_l$ to $v_l'$.*

Now let us go further backwards and observe what happens if one of the current vertices $v_l$ or $v_r$ change. We assume that condition (3.1) holds for path $w$ from $p_1$ to $p_2$ and that $v_l$ changes at $p_2$; see Figure 3.3. The visible left chain is extended by $l_2'$. Nothing changes on the right side of the funnel, and for the left side of the funnel we have

$$\begin{aligned} w \leq K_{\phi_1} l_1 - K_{\phi_2} l_2 &= K_{\phi_1} l_1 - K_{\phi_2} l_2 + K_{\phi_2} l_2' - K_{\phi_2} l_2' \\ &\leq K_{\phi_1}(l_1 + l_2') - K_{\phi_2}(l_2 + l_2')\,. \end{aligned} \tag{3.2}$$

The last inequality holds because $K_\phi = \sqrt{1 + \sin\phi}$ is decreasing with increasing $\phi \in [\frac{\pi}{2}, \pi]$. Here, $l_1 + l_2'$ and $l_2 + l_2'$ are the lengths of the shortest paths from $p_1$ and $p_2$ to $v_l'$, respectively. But (3.2) in fact means that (3.1) remains valid even if changes of $v_l$ or $v_r$ occur.

Under the assumption that (3.1) holds for all small steps where $v_l$ and $v_r$ do not change we can make use of the additivity of (3.1) and obtain the following expression for the path length, $W$, from an initial opening angle $\phi_0$ to the point $p_{end}$ where the line segment $t_l\, t_r$ is reached; see Figure 3.3.

$$W \leq \min\Big( \begin{array}{l} K_{\phi_0}(\text{length of left chain}) - K_\pi l_{end}, \\ K_{\phi_0}(\text{length of right chain}) - K_\pi r_{end} \end{array} \Big).$$

But, since $K_\pi = 1$, this inequality exactly means that we have a competitive factor not larger than $K_{\phi_0}$. Only a curve remains to be found that satisfies (3.1) for small steps.

**3.3. Developing the curve.** One could try to satisfy condition (3.1) by analyzing, for fixed $p_1$, $\phi_1$, and $\phi_2$, which points $p_2$ meet that requirement. To avoid this tedious task, we argue as follows. For fixed $\phi_2$, the point $p_2$ lies on a circular arc $U_{\phi_2}$ through $v_l$ and $v_r$. While $p_2$ moves along the arc $U_{\phi_2}$, the length $l_2$ is strictly increasing while $r_2$ is strictly decreasing. Heuristically, we maximize our chances to satisfy (3.1) if we require that

$$K_{\phi_1} l_1 - K_{\phi_2} l_2 = K_{\phi_1} r_1 - K_{\phi_2} r_2$$

or, equivalently,

$$K_{\phi_2}(l_2 - r_2) = K_{\phi_1}(l_1 - r_1)\,. \tag{3.3}$$

We claim that inside the triangle defined by $\phi_1$ and segments of length $l_1$ and $r_1$ there exists a point $p_2$ on $U_{\phi_2}$ that satisfies (3.3). Indeed, while $p_2$ moves along $U_{\phi_2}$ between the intersections of $U_{\phi_2}$ with the segments of length $l_1$ and $r_1$, the continuous expression $K_{\phi_2}(l_2 - r_2) - K_{\phi_1}(l_1 - r_1)$ changes its sign; see Figure 3.4. If $p_2$ is the intersection of $U_{\phi_2}$ with the segment of length $r_1$, we have $r_2 < r_1$ and $K_{\phi_2} < K_{\phi_1}$, and $K_{\phi_2}(l_2 - r_2) - K_{\phi_1}(l_1 - r_1)$ is positive if $K_{\phi_1} l_1 \le K_{\phi_2} l_2$. Using the law of sine, $K_{\phi_1} l_1 \le K_{\phi_2} l_2$ is equivalent to $\frac{K_{\phi_1}}{\sin\phi_1} \le \frac{K_{\phi_2}}{\sin\phi_2}$. The expression $\frac{K_\phi}{\sin\phi}$ is monotonically increasing for $\frac{\pi}{2} \le \phi < \pi$. For the same reason $K_{\phi_2}(l_2 - r_2) - K_{\phi_1}(l_1 - r_1) < 0$ holds if $p_2$ is the intersection of $U_{\phi_2}$ with the segment of length $l_1$.

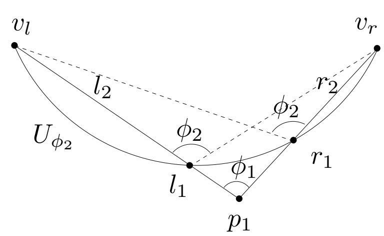


FIG. 3.4. $K_{\phi_2}(l_2 - r_2) - K_{\phi_1}(l_1 - r_1)$ *changes its sign along the circular arc* $U_{\phi_2}$.

Altogether, if we start with the initial values $\phi_0$, $l_0$, $r_0$, we define the fixed constant $A := K_{\phi_0}(l_0 - r_0)$, and for any $\phi_0 \le \phi \le \pi$ with corresponding lengths $l_\phi$ and $r_\phi$ we want that

$$K_\phi(l_\phi - r_\phi) = A \tag{3.4}$$

holds as long as $v_l$ and $v_r$ do not change. In the symmetric case $l_0 = r_0$ this condition means that we walk along the bisector of $v_l$ and $v_r$. Otherwise, condition (3.4) defines a curve which can be determined in the following way. We choose a coordinate system with horizontal axis $v_l\, v_r$, the midpoint being the origin. We scale the coordinate system so that the distance from $v_l$ to $v_r$ equals 1. With this choice we have

$$|A| = |K_\phi(l_\phi - r_\phi)| \le K_\phi = \sqrt{1 + \sin\phi} \tag{3.5}$$

for every $l_\phi$, $r_\phi$, and $K_\phi$ in the triangle defined by $\phi_0$, $l_0$, $r_0$.

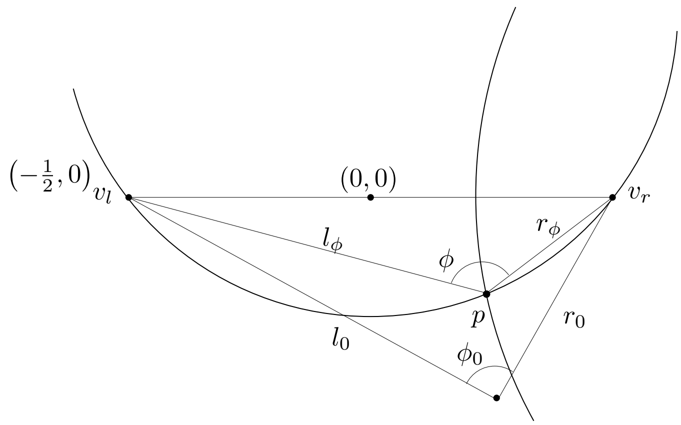


FIG. 3.5. *The right arc of the hyperbola defined by $v_l$, $v_r$, and $(l-r) = \frac{A}{K_\phi}$ and the circle through $v_l$ and $v_r$ defined by angle $\phi$ meet in $p = (X(\phi), Y(\phi))$.*

W.l.o.g. let $l_0 > r_0$. For any $\phi_0 \leq \phi < \pi$ the corresponding point of the curve is the intersection of the hyperbola

$$\frac{X^2}{\left(\frac{A}{2K_\phi}\right)^2} - \frac{Y^2}{\left(\frac{1}{2}\right)^2 - \left(\frac{A}{2K_\phi}\right)^2} = 1 \tag{3.6}$$

with the circle

$$X^2 + \left(Y + \frac{\cot\phi}{2}\right)^2 = \frac{1}{4\sin^2\phi}; \tag{3.7}$$

see Figure 3.5 and the details found in section A.1 of the appendix.

Solving these equations leads us, after some transformations, to the following solutions (for details see section A.2 of the appendix):

$$X(\phi) = \frac{A}{2} \cdot \frac{\cot\frac{\phi}{2}}{1+\sin\phi}\sqrt{\left(1+\tan\frac{\phi}{2}\right)^2 - A^2}, \tag{3.8}$$

$$Y(\phi) = \frac{1}{2}\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi} - 1\right). \tag{3.9}$$

Since $|A| < \sqrt{1+\sin\phi} < 1 + \tan\frac{\phi}{2}$ holds, the functions $X(\phi)$ and $Y(\phi)$ are well defined and continuous while the curve stays below the line segment $v_l v_r$.

Figure 3.6 shows how these curves look for all possible values of $\phi$ and $A$. All points with an initial opening angle of $\frac{\pi}{2}$ lie on the lower half circle.

Two cases can be distinguished. If $|A| < 1$, then the curves can be continuously extended to a point $(\frac{A}{2}, 0)$ on the line segment $v_l\, v_r$. For $|A| > 1$ the curves end up in $v_l$ and $v_r$, respectively, with a limit of opening angles $\phi = \pi - \arcsin(A^2 - 1)$, which satisfies $X(\phi) = \pm\frac{1}{2}$, $Y(\phi) = 0$, and $|A| = K_\phi$. The curves for the limiting cases $|A| = 1$ are emphasized with a thick line in Figure 3.6.

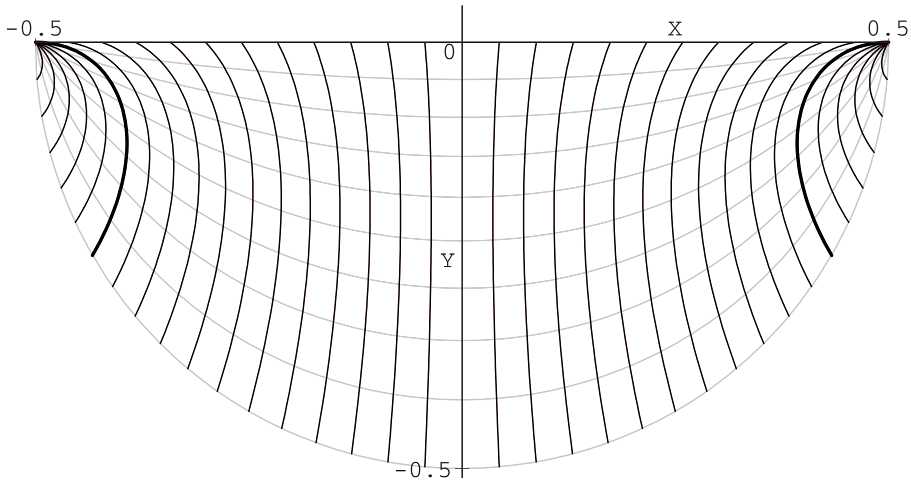


FIG. 3.6. *The curves fulfilling condition* (3.4) *for all values of* $\phi$ *and* $A$.

**3.4. Checking the requirements.** We want to check that the curve defined by (3.8) and (3.9) in section 3.3 satisfies condition (3.1). The arc length of the curve from angle $\phi_1$ to $\phi_2$ has to be compared to the right side of (3.1). Because of (3.3) the min in (3.1) can be dropped.

For $l_0 = r_0$ we trivially have equality in (3.1). W.l.o.g. we can assume $l_0 > r_0$ and $A > 0$. The other case is symmetric. It suffices to check

$$\int_{\phi_1}^{\phi_2} \sqrt{X'(\phi)^2 + Y'(\phi)^2}\, d\phi \le K_{\phi_1} l_{\phi_1} - K_{\phi_2} l_{\phi_2} \qquad \text{for all } \frac{\pi}{2} \le \phi_1 < \phi_2 < \pi\,.$$

Here, $X'(\phi)$ and $Y'(\phi)$ denote the derivatives of $X(\phi)$ and $Y(\phi)$ from (3.8) and (3.9) in $\phi$. The inequality is equivalent to

$$\int_{\phi_1}^{\phi_2} \sqrt{X'(\phi)^2 + Y'(\phi)^2}\, d\phi \le \int_{\phi_1}^{\phi_2} -(K_\phi l_\phi)'\, d\phi \qquad \text{for all } \frac{\pi}{2} \le \phi_1 < \phi_2 < \pi\,, \tag{3.10}$$

since $K_\phi l_\phi$ is a differentiable function in $\phi$. It is sufficient to show that in (3.10) one integrant dominates the other. In the following we will try to simplify this task.

By some transformations (for details see section A.3 of the appendix), we obtain

$$l_\phi = \frac{K_\phi}{A} X(\phi) + \frac{A}{2K_\phi} \quad \text{and therefore} \quad K_\phi l_\phi = \frac{K_\phi^2}{A} X(\phi) + \frac{A}{2}\,. \tag{3.11}$$

Thanks to an idea by Seidel [41] we can make use of the substitution $t = \tan \frac{\phi}{2}$

in $K_\phi$, $X(\phi)$, $Y(\phi)$, and $K_\phi l_\phi$. Thus, we get the functions

$$\begin{aligned}
\overline{K}(t) &:= \sqrt{\frac{(1+t)^2}{1+t^2}},\\
\overline{X}(t) &:= \frac{A}{2t\overline{K}^2(t)}\sqrt{(1+t)^2-A^2} = \frac{A(1+t^2)\sqrt{(1+t)^2-A^2}}{2t(1+t)^2},\\
\overline{Y}(t) &:= \left(\frac{A^2}{\overline{K}^2(t)}-1\right)\frac{1}{2t} = \frac{A^2(1+t^2)-(1+t)^2}{2t(1+t)^2},\\
\overline{Kl}(t) &:= \frac{\overline{K}^2(t)}{A}\overline{X}(t)+\frac{A}{2} = \frac{\sqrt{(1+t)^2-A^2}}{2t}+\frac{A}{2}
\end{aligned}$$

with the identities $\overline{K}(\tan\frac{\phi}{2}) = K_\phi$, $\overline{X}(\tan\frac{\phi}{2}) = X(\phi)$, $\overline{Y}(\tan\frac{\phi}{2}) = Y(\phi)$, and $\overline{Kl}(\tan\frac{\phi}{2}) = K_\phi l_\phi$. The simple identities are proven in section A.4 of the appendix.

We will now simplify (3.10) using terms of the identities above. The left-hand side of (3.10) is equivalent to

$$\begin{aligned}
&\int_{\phi_1}^{\phi_2}\sqrt{\left(\frac{d}{d\phi}\left(\overline{X}\left(\tan(\phi/2)\right)\right)\right)^2+\left(\frac{d}{d\phi}\left(\overline{Y}\left(\tan(\phi/2)\right)\right)\right)^2}\,d\phi\\
&= \int_{\phi_1}^{\phi_2}\frac{d}{d\phi}\left(\tan(\phi/2)\right)\sqrt{\left(\left(\frac{d}{dt}\overline{X}\right)\left(\tan(\phi/2)\right)\right)^2+\left(\left(\frac{d}{dt}\overline{Y}\right)\left(\tan(\phi/2)\right)\right)^2}\,d\phi,
\end{aligned} \tag{3.12}$$

whereas the right-hand side of (3.10) is equivalent to

$$\int_{\phi}^{\phi_2} -\frac{d}{d\phi}\left(\overline{Kl}\left(\tan(\phi/2)\right)\right)\,d\phi = \int_{\phi}^{\phi_2} -\frac{d}{d\phi}\left(\tan(\phi/2)\right)\left(\frac{d}{dt}\overline{Kl}\right)\left(\tan(\phi/2)\right)\,d\phi. \tag{3.13}$$

By applying the rule of substitution to (3.12) and (3.13), (3.10) is equivalent to

$$\int_{\tan\frac{\phi_1}{2}}^{\tan\frac{\phi_2}{2}}\sqrt{\left(\frac{d}{dt}\overline{X}(t)\right)^2+\left(\frac{d}{dt}\overline{Y}(t)\right)^2}\,dt \le \int_{\tan\frac{\phi_1}{2}}^{\tan\frac{\phi_2}{2}} -\frac{d}{dt}\overline{Kl}(t)\,dt\,. \tag{3.14}$$

The function $\tan\frac{\phi}{2}$ is positive, continuous, and monotonically increasing for $\frac{\pi}{2} \le \phi < \pi$. Then it suffices to show that in (3.14) one integrand dominates the other one for every $t$ in the integration interval $[\tan\frac{\phi_1}{2}, \tan\frac{\phi_2}{2}]$ for all $\frac{\pi}{2} \le \phi_1 < \phi_2 < \pi$. We make use of the following facts. For every $t \in [\tan\frac{\phi_1}{2}, \tan\frac{\phi_2}{2}]$ there is always a unique $\phi \in [\phi_1, \phi_2]$ with $t = \tan\frac{\phi}{2}$. Additionally we can assume $A < \sqrt{1+\sin\phi}$ from (3.5).

Altogether, it suffices to prove that

$$\sqrt{\overline{X}'(t)^2+\overline{Y}'(t)^2} \le -\overline{Kl}'(t) \tag{3.15}$$

holds for $t = \tan\frac{\phi}{2}$, $\frac{\pi}{2} \le \phi < \pi$, and $0 < A < \sqrt{1+\sin\phi}$. Here for convenience $\overline{X}'(t)$, $\overline{Y}'(t)$, and $\overline{Kl}'(t)$ denote the derivatives of the corresponding functions in $t$. We insert the following identities (see section A.5 of the appendix for details) into (3.15).

$$-\overline{Kl}'(t) = \frac{(1+t)-A^2}{2t^2\sqrt{(1+t)^2-A^2}},$$

$$\overline{X}'(t) = \frac{A\left(A^2\left((1+t)^3 - 4t^2\right) - 1 - 4t - 4t^2 + t^4\right)}{2(1+t)^3 t^2 \sqrt{(1+t)^2 - A^2}},$$

$$\overline{Y}'(t) = \frac{(1+t)^3 - A^2\left((1+t)^3 - 4t^2\right)}{2(1+t)^3 t^2} .$$

Note that $-\overline{Kl}'(t) > 0$ follows from $t \geq \tan\frac{\pi}{4} = 1$ and $A^2 < 2$; see (3.5). Thus, after squaring, the following remains to be shown:

$$F(t,A) \geq 0 \text{ for all } \quad t = \tan\frac{\phi}{2}, \ \ \frac{\pi}{2} \leq \phi < \pi, \quad \text{and} \quad 0 < A < \sqrt{1+\sin\phi} \quad \text{where}$$

$$\begin{aligned} F(t,A) &:= \left(\overline{Kl}'(t)\right)^2 - \left(\overline{X}'(t)^2 + \overline{Y}'(t)^2\right) \\ &= \frac{-(t-1)A^2\left((A^2-1)(t^2+3) - 4t\right)}{4(1+t)^3 t^2\left((1+t)^2 - A^2\right)}. \end{aligned}$$

The last inequality is proven in section A.6 of the appendix. The denominator of $F(t,A)$ is positive since $t \geq \tan\frac{\pi}{4} = 1$ and $A^2 < 2$ holds; see (3.5). Therefore it suffices to show that

$$(A^2-1)(t^2+3) - 4t \leq 0 . \tag{3.16}$$

We minimize our chances to satisfy (3.16) if $A$ achieves a maximal value greater than 1. Substituting $A^2$ by $1+\sin(\phi)$, the greatest possible value for $A^2$, and $t$ by $\tan\frac{\phi}{2}$, the inequality (3.16) holds if

$$2\cos\phi\tan\frac{\phi}{2} \leq 0 .$$

The details are given in section A.7 of the appendix. The last inequality holds for $\frac{\pi}{2} \leq \phi < \pi$. This proves (3.15) and therefore (3.1) for the curves of section 3.3.

**3.5. Opening angles below 90°.** So far we have seen that there is a strategy that is competitive with factor $\sqrt{2}$ for opening angles greater than or equal to $\frac{\pi}{2}$. There are already methods to accomplish the task for funnels with opening angles running from an initial angle $\phi_0 < \frac{\pi}{2}$ to an opening angle of $\frac{\pi}{2}$. As was already shown by Semrau [42] and also in López-Ortiz [28], any strategy which achieves a factor $\geq \sqrt{2}$ for all funnels with $\phi_0 \geq \frac{\pi}{2}$ can be adapted to the general case without changing its factor. They suggest a walk along the fixed angular bisector of the current pair $v_l$ and $v_r$ until an opening angle of $\frac{\pi}{2}$ is reached. If the opening angle of $\frac{\pi}{2}$ is reached, one can proceed, for example, with the strategy given in section 3.3. So we are done here.

In the following we show that our idea is universal, and for completeness we consider the case $0 < \phi < \frac{\pi}{2}$ analogously. Looking backwards as in section 3.2 we can assume that there is a strategy which is competitive with factor $\sqrt{2}$ starting at point $p_2$ with a opening angle $\frac{\pi}{2} \geq \phi_2 > 0$. Again we want to extend the strategy to initial opening angles $\phi_1$ at starting points $p_1$ with $\frac{\pi}{2} \geq \phi_2 > \phi_1 > 0$; see again Figure 3.2. The only difference to the former consideration is that the factor need not vary any longer with respect to the opening angle. The worst-case factor of $\sqrt{2}$ is already in use, and we want to achieve this factor when starting at $p_1$.

Thus, with the same arguments and notation as in section 3.2, it suffices to show that there is a strategy so that

$$w \leq \min(\sqrt{2}\, l_1 - \sqrt{2}\, l_2, \sqrt{2}\, r_1 - \sqrt{2}\, r_2) \tag{3.17}$$

holds between the changes of $v_l$ and $v_r$ as long as the opening angle is smaller than $\frac{\pi}{2}$.

Again, similar to section 3.3, we want to satisfy (3.17) and therefore require that

$$\sqrt{2}\,(l_1 - l_2) = \sqrt{2}\,(r_1 - r_2) \text{ or, equivalently, } (l_2 - r_2) = (l_1 - r_1)\,. \tag{3.18}$$

We consider two cases: For $l_0 = r_0$ we follow the fixed angular bisector in the triangle defined by $l_0$, $r_0$, and $\phi_0$. In this case, as already stated in the beginning of section 3.4 with $l_0 = r_0$, the equality

$$w = K_{\phi_1} l_1 - K_{\phi_2} l_2 = K_{\phi_1} r_1 - K_{\phi_2} r_2$$

holds. Then (3.17) follows from $K_{\phi_1} \leq K_{\phi_2} \leq \sqrt{2}$ for $\frac{\pi}{2} \geq \phi_2 > \phi_1 > 0$.

If we start with an initial difference $1 > D := (l_0 - r_0) > 0$ at point $s$, (3.18) means that we follow the current angular bisector (CAB) of $v_l$ and $v_r$, and the resulting curve is a hyperbola through $s$ with fix-points $v_l$ and $v_r$; see Icking, Klein, and Langetepe [20]. For the street problem the strategy CAB was already successfully analyzed for small angles in López-Ortiz and Schuierer [31]. We show that our approach works as well. The transformations of sections 3.3 and 3.4 become much easier since the constant does not depend on $\phi$, and all transformations are presented in detail in section A.8 of the appendix. We proceed as before and obtain the coordinates

$$\begin{aligned} X(\phi) &= \frac{D}{2} \cot \frac{\phi}{2} \sqrt{\left(1 + \tan^2 \frac{\phi}{2}\right) - D^2}, \\ Y(\phi) &= \frac{1}{2} \cot \frac{\phi}{2} (D^2 - 1). \end{aligned}$$

Now $K_\phi l_\phi$ simplifies to

$$\sqrt{2}\, l_\phi = \frac{\sqrt{2}}{D} X(\phi) + \frac{D}{\sqrt{2}}\,.$$

In this case there is no need to simplify the terms by a substitution. In analogy to the previous section it suffices to prove that in (3.10) one integrand dominates the other one; that is,

$$\sqrt{X'(\phi)^2 + Y'(\phi)^2} \leq -(K_\phi l_\phi)' \tag{3.19}$$

for all $\frac{\pi}{2} \geq \phi > 0$ and $0 < D < 1$. Altogether it suffices to show

$$\begin{aligned} F(\phi, D) &\geq 0 \quad \text{for all} \quad 0 < \phi \leq \frac{\pi}{2} \quad \text{and} \quad 0 < D < 1, \quad \text{where} \\ F(\phi, D) &:= X'(\phi)^2 \left(\frac{2}{D^2} - 1\right) - Y'(\phi)^2 \\ &= \frac{(D^2 - 1)^2 \left(1 - \tan^2 \frac{\phi}{2}\right)}{4(\cos\phi - 1)^2 \left(\left(1 + \tan^2 \frac{\phi}{2}\right) - D^2\right)}. \end{aligned}$$

It remains to be shown that

$$1 - \tan^2 \frac{\phi}{2} \geq 0 \quad \text{and} \quad 1 + \tan^2 \frac{\phi}{2} - D^2 \geq 0$$

holds, which follows from $D^2 < 1$ and $\tan \frac{\phi}{2} \leq 1$ for $\frac{\pi}{2} \geq \phi > 0$. See section A.8 of the appendix for all details.

**3.6. The main result.** To summarize, our strategy for searching a goal in an unknown street works as follows.

*Strategy WCA (worst-case aware).* If the initial opening angle is less than $\frac{\pi}{2}$, walk along the current angular bisector of $v_l$ and $v_r$ until a right opening angle is reached.
Depending on the actual parameters $\phi_0$, $l_0$, and $r_0$, walk along the corresponding curve from section 3.3 until one of $v_l$ and $v_r$ changes. Switch over to the curve corresponding to the new parameters $\phi_1$, $l_1$, and $r_1$. Continue until the line $t_l\, t_r$ is reached.

THEOREM 3.2. *By using strategy* WCA *we can search a goal in an unknown street with a competitive factor of* $\sqrt{2}$ *at the most. This is optimal.*

The proof can be found in sections 3.1 through 3.5. In Figure 3.7 a complete path of *WCA* inside a street is shown.

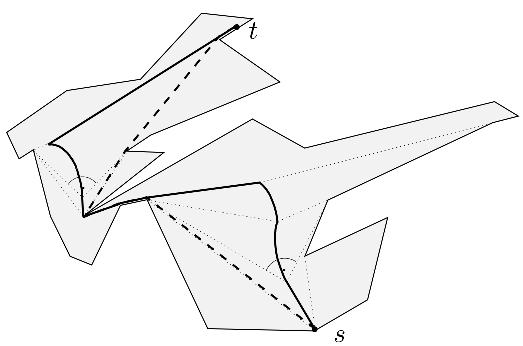


FIG. 3.7. *A street and the path generated by WCA.*

**4. Conclusions.** We have developed a competitive strategy for walking in streets which guarantees an optimal factor of $\sqrt{2}$ at the most in the worst case, thereby settling an old open problem. Furthermore, the strategy is even better for an initial opening angle $\phi_0 > \frac{\pi}{2}$, in which case an optimal factor $K_{\phi_0} = \sqrt{1 + \sin \phi_0}$ between 1 and $\sqrt{2}$ is achieved.

It would be interesting to see if there are substantially different but also optimal strategies.

**Appendix. Formal calculations.** This appendix contains the formal calculations needed in the main text. They are presented in a resolution that makes it possible to follow step by step. Nevertheless, the reader might prefer to enter start and target formulae into some math-tool, for example, Maple or Mathematica, and have correctness checked automatically.

**A.1. Definition of the circle and the hyperbola.** We choose a coordinate system with horizontal axis $v_l\,v_r$, the midpoint being the origin. We scale the coordinate system so that the distance from $v_l$ to $v_r$ equals 1. Let $p$ be the point at a fixed opening angle $\phi$ on the curve we want to construct. Then two constraints must be met. First, the difference $l(p)-r(p)$ of the length from $p$ to $v_l$ and $v_r$, correspondingly, must equal $\frac{A}{K_\phi}$. The locus of all such points $p$ is a hyperbola. Second, $p$ sees $v_l$ and $v_r$ at the angle $\phi$. The locus of all these points $p$ is a circle; see Figure A.1.

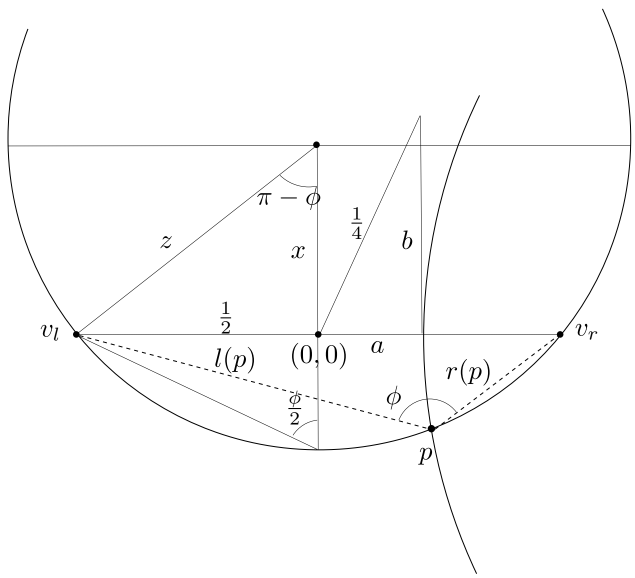


FIG. A.1. *The right arc of the hyperbola defined by $v_l$, $v_r$, and $(l(p) - r(p)) = \frac{A}{K_\phi}$ and the circle through $v_l$ and $v_r$ defined by angle $\phi$.*

The hyperbola reads

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \ ,$$

where $2a = (l(p) - r(p)) = \frac{A}{K_\phi}$ and $b^2 + a^2 = e^2 = \frac{1}{4}$ hold. So we have $a^2 = (\frac{A}{2K_\phi})^2$ and $b^2 = \frac{1}{4} - (\frac{A}{2K_\phi})^2$. The circle is defined by

$$X^2 + (Y - x)^2 = z^2 \ . \tag{A.1}$$

It remains to compute the parameters of the circle, $x$ and $z$. From the law of sine we get

$$\frac{z}{\sin\frac{\pi}{2}} = \frac{1}{2\sin(\pi-\phi)} = \frac{1}{2\sin\phi},$$
$$\frac{z-x}{\sin\left(\pi - \frac{\pi}{2} - \frac{\phi}{2}\right)} = \frac{z-x}{\cos\frac{\phi}{2}} = \frac{1}{2\sin\frac{\phi}{2}},$$

and therefore $z = \frac{1}{2\sin\phi}$ and

$$x = z - \frac{1}{2}\cot\frac{\phi}{2} = \frac{1}{2\sin\phi} - \frac{1}{2}\cot\frac{\phi}{2} = \frac{1 - 2\cos^2\frac{\phi}{2}}{4\sin\frac{\phi}{2}\cos\frac{\phi}{2}} = -\frac{\cot\phi}{2} \ .$$

**A.2. Intersection of the circle and the hyperbola.** In order to verify the expressions

$$X(\phi) = \frac{A}{2} \cdot \frac{\cot\frac{\phi}{2}}{1+\sin\phi}\sqrt{\left(1+\tan\frac{\phi}{2}\right)^2 - A^2}, \tag{A.2}$$

$$Y(\phi) = \frac{1}{2}\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi} - 1\right), \tag{A.3}$$

we insert them into the equations

$$\frac{X^2}{\left(\frac{A}{2K_\phi}\right)^2} - \frac{Y^2}{\left(\frac{1}{2}\right)^2 - \left(\frac{A}{2K_\phi}\right)^2} = 1, \tag{A.4}$$

$$X^2 + \left(Y + \frac{\cot\phi}{2}\right)^2 = \frac{1}{4\sin^2\phi}. \tag{A.5}$$

For (A.4) we have

$$\frac{\left(\frac{A}{2}\cdot\frac{\cot\frac{\phi}{2}}{1+\sin\phi}\sqrt{\left(1+\tan\frac{\phi}{2}\right)^2 - A^2}\right)^2}{\left(\frac{A}{2K_\phi}\right)^2} - \frac{\left(\frac{1}{2}\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi}-1\right)\right)^2}{\left(\frac{1}{2}\right)^2 - \left(\frac{A}{2K_\phi}\right)^2}$$
$$= \left(\frac{\cot\frac{\phi}{2}}{K_\phi}\right)^2\left(\left(1+\tan\frac{\phi}{2}\right)^2 - A^2\right) - \frac{\cot^2\frac{\phi}{2}\left(\left(\frac{A}{K_\phi}\right)^2 - 1\right)^2}{1-\left(\frac{A}{K_\phi}\right)^2}$$
$$= \left(\frac{\cot\frac{\phi}{2}}{K_\phi}\right)^2\left(\left(1+\tan\frac{\phi}{2}\right)^2 - A^2\right) + \cot^2\frac{\phi}{2}\left(\left(\frac{A}{K_\phi}\right)^2 - 1\right)$$
$$= \cot^2\frac{\phi}{2}\left(\frac{\left(1+\tan\frac{\phi}{2}\right)^2}{1+\sin\phi} - 1\right) = 1\ .$$

The conclusion is true since the identity

$$1+\sin\phi = 1 + \frac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{\left(1+\tan\frac{\phi}{2}\right)^2}{1+\tan^2\frac{\phi}{2}} \tag{A.6}$$

holds.

For proving (A.5) we argue as follows:

$$\left(\frac{A}{2}\cdot\frac{\cot\frac{\phi}{2}}{1+\sin\phi}\sqrt{\left(1+\tan\frac{\phi}{2}\right)^2 - A^2}\right)^2 + \left(\frac{1}{2}\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi}-1\right) + \frac{\cot\phi}{2}\right)^2$$
$$= \left(\frac{A}{2}\cdot\frac{\cot\frac{\phi}{2}}{1+\sin\phi}\right)^2\left(\left(1+\tan\frac{\phi}{2}\right)^2 - A^2\right)$$

$$
\begin{aligned}
&+\left(\frac{1}{2}\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi}-1\right)\right)^2+\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi}-1\right)\frac{\cot\phi}{2}+\left(\frac{\cot\phi}{2}\right)^2\\
=&\left(\frac{A}{2}\cdot\frac{\cot\frac{\phi}{2}}{1+\sin\phi}\right)^2\left(1+\tan\frac{\phi}{2}\right)^2\\
&+\left(\frac{1}{2}\cot\frac{\phi}{2}\right)^2\left(-2\frac{A^2}{1+\sin\phi}+1\right)+\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi}-1\right)\frac{\cot\phi}{2}+\left(\frac{\cot\phi}{2}\right)^2\\
=&\left(\frac{\cot\frac{\phi}{2}}{2}-\frac{\cot\phi}{2}\right)^2+\frac{A^2\cot^2\frac{\phi}{2}}{4(1+\sin\phi)}\left(\frac{\left(1+\tan\frac{\phi}{2}\right)^2}{1+\sin\phi}-2+2\frac{\cot\phi}{\cot\frac{\phi}{2}}\right)\\
=&\frac{1}{4\sin^2\phi}+\frac{A^2\cot^2\frac{\phi}{2}}{4(1+\sin\phi)}\left(\tan^2\frac{\phi}{2}+1-2+\frac{1-\tan^2\frac{\phi}{2}}{\tan\frac{\phi}{2}}\tan\frac{\phi}{2}\right)\\
=&\frac{1}{4\sin^2\phi}+\frac{A^2\cot^2\frac{\phi}{2}}{4(1+\sin\phi)}\cdot 0=\frac{1}{4\sin^2\phi}\,.
\end{aligned}
$$

Here we made use of (A.6) and the identities

$$\left(\frac{\cot\frac{\phi}{2}}{2}-\frac{\cot\phi}{2}\right)^2=\frac{1}{4}\left(\frac{\sin\phi}{1-\cos\phi}-\frac{\cos\phi}{\sin\phi}\right)^2=\frac{1}{4}\frac{1}{\sin^2\phi}$$

and

$$\cot\phi=\frac{1-\tan^2\frac{\phi}{2}}{2\tan\frac{\phi}{2}}\,.$$

**A.3. Representation of $l_\phi$.** Considering the hyperbola (see Figure 3.5), we have

$$
\begin{aligned}
l_\phi &= \sqrt{\left(X(\phi)+\frac{1}{2}\right)^2+Y^2(\phi)}\\
&\overset{(3.6)}{=} \sqrt{\left(X(\phi)+\frac{1}{2}\right)^2-X^2(\phi)-\left(\frac{1}{2}\right)^2+\left(\frac{A}{2K_\phi}\right)^2+\frac{\left(\frac{1}{2}\right)^2}{\left(\frac{A}{2K_\phi}\right)^2}X^2(\phi)}\\
&= \sqrt{\left(\frac{K_\phi}{A}\right)^2X^2(\phi)+X(\phi)+\left(\frac{A}{2K_\phi}\right)^2}\\
&= \sqrt{\left(\frac{K_\phi}{A}X(\phi)+\frac{A}{2K_\phi}\right)^2}=\frac{K_\phi}{A}X(\phi)+\frac{A}{2K_\phi}\,.
\end{aligned}
$$

**A.4. Applying the substitution $t = \tan\frac{\phi}{2}$.** Applying (A.6) and $t=\tan\frac{\phi}{2}$ we have

$$K_\phi=\sqrt{1+\sin\phi}=\sqrt{\frac{(1+t)^2}{1+t^2}}=:\overline{K}(t)\,. \tag{A.7}$$

A straightforward application of (A.7) and $t = \tan\frac{\phi}{2}$ to (A.2) and (A.3) leads to

$$X(\phi) = \frac{A}{2t\overline{K}^2(t)}\sqrt{(1+t)^2 - A^2} = \frac{A(1+t^2)\sqrt{(1+t)^2-A^2}}{2t(1+t)^2} =: \overline{X}(t)\ ,$$

$$Y(\phi) = \left(\frac{A^2}{\overline{K}^2(t)} - 1\right)\frac{1}{2t} = \frac{A^2(1+t^2) - (1+t)^2}{2t(1+t)^2} =: \overline{Y}(t)\ .$$

Using the representations of $\overline{X}(t)$ and $\overline{K}(t)$ we conclude that

$$K_\phi l_\phi = \frac{K_\phi}{A}X(\phi) + \frac{A}{2K_\phi} = \frac{\overline{K}^2(t)}{A}\overline{X}(t) + \frac{A}{2} = \frac{\sqrt{(1+t)^2 - A^2}}{2t} + \frac{A}{2} =: \overline{Kl}(t)$$

holds.

**A.5. Computing the derivatives in *t*.** We apply simple derivation rules:

$$\begin{aligned}
-\overline{Kl}'(t) &= -\left(\frac{(1+t)}{2t\sqrt{(1+t)^2-A^2}} - \frac{\sqrt{(1+t)^2-A^2}}{2t^2}\right)\\
&= -\left(\frac{t(1+t) - (1+t)^2 + A^2}{2t^2\sqrt{(1+t)^2-A^2}}\right)\\
&= \frac{(1+t) - A^2}{2t^2\sqrt{(1+t)^2-A^2}},
\end{aligned}$$

$$\begin{aligned}
\overline{Y}'(t) &= \left(\frac{A^2(1+t^2)}{2t(1+t)^2} - \frac{1}{2t}\right)'\\
&= \frac{2tA^2}{2t(1+t)^2} + A^2(1+t^2)\left(-\frac{(1+t)^2 + 2(1+t)t}{2(1+t)^4t^2}\right) + \frac{1}{2t^2}\\
&= \frac{(1+t)^3 - A^2\left(-2t^2(1+t) + (1+t^2)(1+3t)\right)}{2(1+t)^3t^2}\\
&= \frac{(1+t)^3 - A^2\left((1+t)^3 - 4t^2\right)}{2(1+t)^3t^2},
\end{aligned}$$

$$\begin{aligned}
\overline{X}'(t) &= \left(\frac{A(1+t^2)}{2t(1+t)^2}\sqrt{(1+t)^2-A^2}\right)'\\
&= \left(\frac{2tA}{2t(1+t)^2} + A(1+t^2)\left(-\frac{(1+t)^2 + 2(1+t)t}{2(1+t)^4t^2}\right)\right)\sqrt{(1+t)^2-A^2}\\
&\quad + \frac{1+t}{\sqrt{(1+t)^2-A^2}}\frac{A(1+t^2)}{2t(1+t)^2}\\
&= \frac{-A((1+t)^3 - 4t^2)}{2(1+t)^3t^2\sqrt{(1+t)^2-A^2}}((1+t)^2 - A^2) + \frac{t(1+t)^2A(1+t^2)}{2(1+t)^3t^2\sqrt{(1+t)^2-A^2}}
\end{aligned}$$

$$= \frac{-A((1+t)^3-4t^2)((1+t)^2-A^2)+t(1+t)^2A(1+t^2)}{2(1+t)^3t^2\sqrt{(1+t)^2-A^2}}$$

$$= \frac{A(A^2\left((1+t)^3-4t^2\right))-A\left(((1+t)^3-4t^2)(1+t)^2-t(1+t)^2(1+t^2)\right)}{2(1+t)^3t^2\sqrt{(1+t)^2-A^2}}$$

$$= \frac{A(A^2\left((1+t)^3-4t^2\right))-A((1+t)^5-(1+t)^2(4t^2+t+t^3))}{2(1+t)^3t^2\sqrt{(1+t)^2-A^2}}$$

$$= \frac{A(A^2\left((1+t)^3-4t^2\right))-A((1+t)^5-(1+2t+t^2)(4t^2+t+t^3))}{2(1+t)^3t^2\sqrt{(1+t)^2-A^2}}$$

$$= \frac{A(A^2\left((1+t)^3-4t^2\right))-A((1+t)^5-t-6t^2-10t^3-6t^4-t^5)}{2(1+t)^3t^2\sqrt{(1+t)^2-A^2}}$$

$$= \frac{A\left(A^2\left((1+t)^3-4t^2\right)-1-4t-4t^2+t^4\right)}{2(1+t)^3t^2\sqrt{(1+t)^2-A^2}}.$$

**A.6. How to get $F(t, A)$.** We will make use of the following identities:

$$2(t^2-2t-1)-(4t^2-(1+t)^3)+2(1+t) = (1+t)(t^2+1), \tag{A.8}$$

$$(t^2-2t-1)(1+t)^2 = (t^4-1-4t-4t^2), \tag{A.9}$$

$$\begin{aligned}(t-1)(t+3)t^2 &= -3t^2+2t^3+t^4\\ &= -2(1+t)^3-(t^2-2t-1)^2\\ &\quad +(1+t)^2-8t^2(1+t)+2(1+t)^4,\end{aligned} \tag{A.10}$$

$$(1+t)^3+(4t^2-(1+t)^3)(t^2+1) = t^2(1-t)(t^2+3)\ . \tag{A.11}$$

$$F(t,A) := \overline{Kl}'(t)^2-\overline{X}'(t)^2-\overline{Y}'(t)^2$$

$$= \left(\frac{(1+t)-A^2}{2t^2\sqrt{(1+t)^2-A^2}}\right)^2-\left(\frac{A\left(A^2\left((1+t)^3-4t^2\right)-1-4t-4t^2+t^4\right)}{2(1+t)^3t^2\sqrt{(1+t)^2-A^2}}\right)^2$$

$$-\left(\frac{(1+t)^3-A^2\left((1+t)^3-4t^2\right)}{2(1+t)^3t^2}\right)^2$$

$$= \frac{\left(((1+t)-A^2)(1+t)^3\right)^2}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}-\frac{\left(A\left(A^2\left((1+t)^3-4t^2\right)-1-4t-4t^2+t^4\right)\right)^2}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}$$

$$-\frac{\left((1+t)^3-A^2\left((1+t)^3-4t^2\right)\right)^2\left((1+t)^2-A^2\right)}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}$$

$$= \frac{(1+t)^6A^4-2(1+t)^7A^2+(1+t)^8}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}$$

$$
\begin{aligned}
&-\frac{((1+t)^3-4t^2)^2A^6+2(t^4-1-4t-4t^2)((1+t)^3-4t^2)A^4}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}\\
&\qquad\qquad -\frac{(t^4-1-4t-4t^2)^2A^2}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}\\
&-\frac{-((1+t)^3-4t^2)^2A^6+(-2(1+t)^3(4t^2-(1+t)^3)+(4t^2-(1+t)^3)^2(1+t)^2)A^4}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}\\
&\qquad\qquad -\frac{(-(1+t)^6+2(1+t)^5(4t^2-(1+t)^3))A^2+(1+t)^8}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}\\
\stackrel{(A.9)}{=}\ & A^4\frac{(1+t)^6+2(t^2-2t-1)(1+t)^2(4t^2-(1+t)^3)}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}\\
&+A^4\frac{-(-2(1+t)^3(4t^2-(1+t)^3)+(4t^2-(1+t)^3)^2(1+t)^2)}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}\\
&+A^2\frac{-2(1+t)^7-(t^2-2t-1)^2(1+t)^4-(-(1+t)^6+2(1+t)^5(4t^2-(1+t)^3))}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}\\
=\ & A^4\frac{(1+t)^6+(1+t)^2(4t^2-(1+t)^3)(2(t^2-2t-1)-(4t^2-(1+t)^3)+2(1+t))}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}\\
&+A^2\frac{-2(1+t)^7-(1+t)^4(t^2-2t-1)^2+(1+t)^6-2(1+t)^5(4t^2-(1+t)^3))}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}\\
\stackrel{(A.8)}{=}\ & A^4\frac{(1+t)^6+(1+t)^2(4t^2-(1+t)^3)(1+t)(t^2+1)}{\left(2(1+t)^3t^2\right)^2((1+t)^2-A^2)}\\
&+A^2\frac{(1+t)\left(-2(1+t)^3-(t^2-2t-1)^2+(1+t)^2-8t^2(1+t)+2(1+t)^4\right)}{(1+t)^3(2t^2)^2((1+t)^2-A^2)}\\
\stackrel{(A.10)}{=}\ & A^4\frac{(1+t)^3+(4t^2-(1+t)^3)(t^2+1)}{(1+t)^3(2t^2)^2((1+t)^2-A^2)}+A^2\frac{(1+t)(t-1)(t+3)t^2}{(1+t)^3(2t^2)^2((1+t)^2-A^2)}\\
\stackrel{(A.11)}{=}\ & A^4\frac{-t^2(t-1)(t^2+3)}{(1+t)^3(2t^2)^2((1+t)^2-A^2)}+A^2\frac{(1+t)(t-1)(t+3)}{4t^2(1+t)^3((1+t)^2-A^2)}\\
=\ & \frac{-(t-1)A^2\left((A^2-1)(t^2+3)-4t\right)}{4(1+t)^3t^2((1+t)^2-A^2)}.
\end{aligned}
$$

**A.7. Transformation of the nominator of $F(t, A)$.**

$$
\sin\phi\left(\tan^2\frac{\phi}{2}+3\right)-4\tan\frac{\phi}{2}=\sin\phi\left(\frac{2\tan\frac{\phi}{2}}{\sin\phi}+2\right)-4\tan\frac{\phi}{2}
$$

$$
\begin{aligned}
&= 2\sin\phi - 2\tan\frac{\phi}{2} \\
&= 2\sin\frac{\phi}{2}\left(2\cos\frac{\phi}{2} - \frac{1}{\cos\frac{\phi}{2}}\right) \\
&= 2\sin\frac{\phi}{2}\left(\frac{2\cos^2\frac{\phi}{2} - 1}{\cos\frac{\phi}{2}}\right) \\
&= 2\sin\frac{\phi}{2}\left(\frac{\cos\phi}{\cos\frac{\phi}{2}}\right) = 2\cos\phi\tan\frac{\phi}{2}.
\end{aligned}
$$

**A.8. The simple case $\boldsymbol{\phi < \frac{\pi}{2}}$.** In order to obtain

$$
\begin{aligned}
X(\phi) &= \frac{D}{2}\cot\frac{\phi}{2}\sqrt{\left(1+\tan^2\frac{\phi}{2}\right) - D^2}, \\
Y(\phi) &= \frac{1}{2}\cot\frac{\phi}{2}(D^2 - 1),
\end{aligned}
$$

and

$$
K_\phi l_\phi := \sqrt{2}\, l_\phi = \frac{\sqrt{2}}{D}X(\phi) + \frac{D}{\sqrt{2}},
$$

we simply replace $\frac{A}{K_\phi}$ by $D$ in (3.8), (3.9), and (3.11). For (3.8), which corresponds to $X(\phi)$, we have to make use of identity (A.6).

Since we did not make use of a substitution here, the derivatives in $\phi$ are given as follows:

$$
\begin{aligned}
-(K_\phi l_\phi)' &= -\frac{\sqrt{2}}{D}X'(\phi), \\
Y'(\phi) &= -\frac{1}{4\sin^2\frac{\phi}{2}}(D^2 - 1) \\
&= \frac{(D^2 - 1)}{2(\cos\phi - 1)}, \\
X'(\phi) &= \left(\frac{D}{2}\cot\frac{\phi}{2}\sqrt{\left(1+\tan^2\frac{\phi}{2}\right) - D^2}\right)' \\
&= \frac{D}{2(\cos\phi - 1)}\sqrt{\left(1+\tan^2\frac{\phi}{2}\right) - D^2} + \frac{D}{2}\cot\frac{\phi}{2}\left(\frac{-\tan\frac{\phi}{2}\frac{1}{\cos^2\frac{\phi}{2}}}{2\sqrt{\left(1+\tan^2\frac{\phi}{2}\right) - D^2}}\right) \\
&= \frac{\frac{D}{(\cos\phi-1)}\left(\left(1+\tan^2\frac{\phi}{2}\right) - D^2\right) - \frac{D}{2}\frac{1}{\cos^2\frac{\phi}{2}}}{2\sqrt{\left(1+\tan^2\frac{\phi}{2}\right) - D^2}} \\
&= \frac{\frac{D(1-D^2)}{(\cos\phi-1)} + \frac{D\tan^2\frac{\phi}{2}}{(\cos\phi-1)} - \frac{D}{2}\frac{1}{\cos^2\frac{\phi}{2}}}{2\sqrt{\left(1+\tan^2\frac{\phi}{2}\right) - D^2}}
\end{aligned}
$$

$$= \frac{D(1-D^2)}{2(\cos\phi - 1)\sqrt{\left(1+\tan^2\frac{\phi}{2}\right) - D^2}}\,.$$

It remains to compute $F(\phi, D)$.

$$\begin{aligned}
F(\phi, D) &:= (K_\phi l_\phi)^2 - (X'(\phi))^2 - (Y'(\phi))^2 \\
&= X'(\phi)^2\left(\frac{2}{D^2} - 1\right) - Y'(\phi)^2 \\
&= \frac{D^2(1-D^2)^2\left(\frac{2}{D^2}-1\right)}{4(\cos\phi-1)^2\left(\left(1+\tan^2\frac{\phi}{2}\right)-D^2\right)} - \frac{(D^2-1)^2}{4(\cos\phi-1)^2} \\
&= \frac{(D^2-1)^2\left((2-D^2) - \left(\left(1+\tan^2\frac{\phi}{2}\right)-D^2\right)\right)}{4(\cos\phi-1)^2\left(\left(1+\tan^2\frac{\phi}{2}\right)-D^2\right)} \\
&= \frac{(D^2-1)^2\left(1-\tan^2\frac{\phi}{2}\right)}{4(\cos\phi-1)^2\left(\left(1+\tan^2\frac{\phi}{2}\right)-D^2\right)}.
\end{aligned}$$

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