Voronoi Diagram and Delaunay Triangulation

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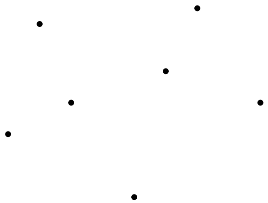
Outline

1. Voronoi Diagrams and Delaunay Triangulations
   - Properties and Duality
2. 3D geometric transformation
3. Applications

Voronoi Diagram

- Given a set S of n point sites, Voronoi Diagram $V(S)$ is a planar subdivision



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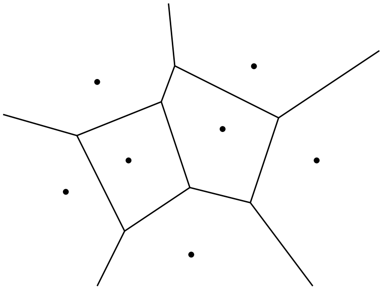

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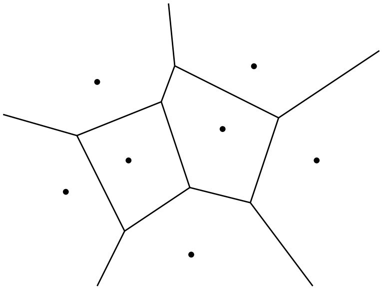

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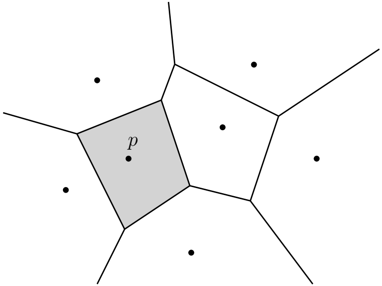

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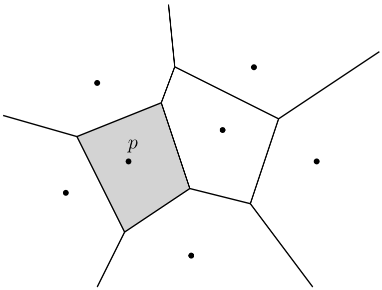

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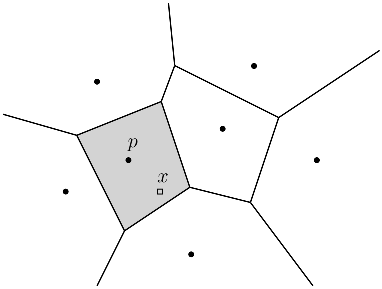

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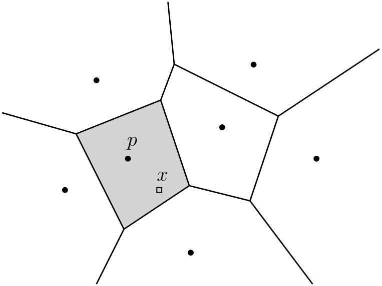
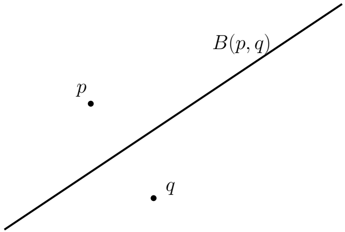
  2. For each point $x \in \mathrm{VR}(p, S)$, p is its closest site in S.
- $\mathrm{VR}(p, S)$ is the locus of points closer to p than any other site.



Voronoi Region

- Bisector $\boldsymbol{B(p,q)} = \{x \in R^2 \mid d(x,p) = d(x,q)\}$.



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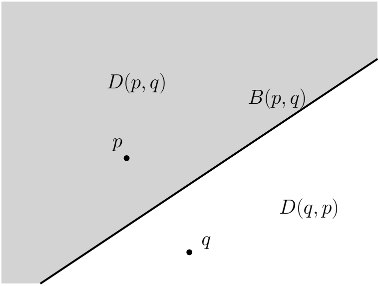
  - Two half-planes $D(p,q)$ and $D(q,p)$ separated by $B(p,q)$.



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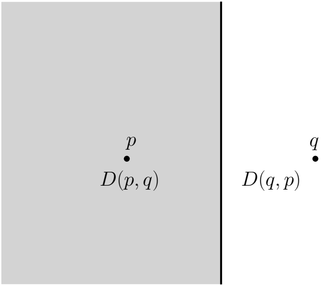

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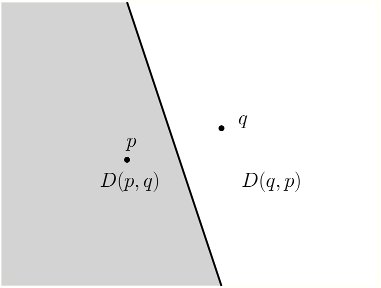

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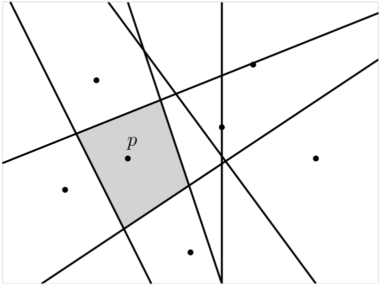
  - Two half-planes $D(p,q)$ and $D(q,p)$ separated by $B(p,q)$.
- $$\mathrm{VR}(p,S) = \bigcap_{q \in S, q \neq p} D(p,q).$$



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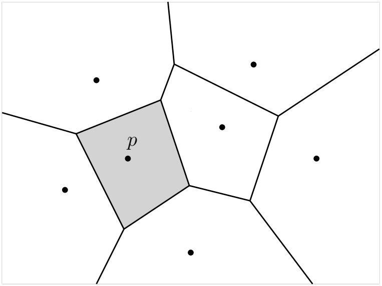

Voronoi Edge and Vertex

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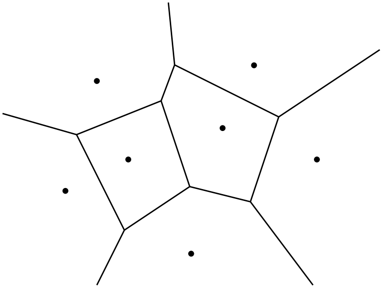
  - Common intersection between two adjacent Voronoi regions $\mathrm{VR}(p, S)$ and $\mathrm{VR}(q, S)$



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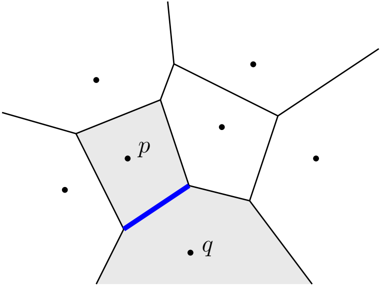

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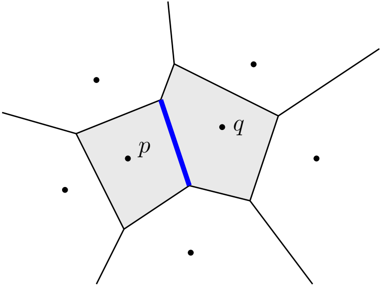

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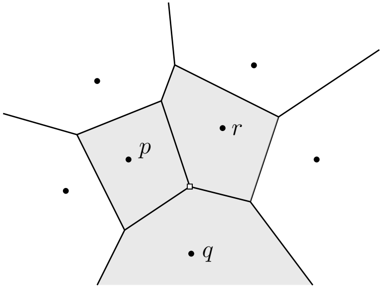
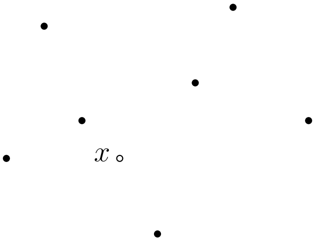
- Voronoi Vertex
  - Common intersection among more than two Voronoi regions $\mathrm{VR}(p, S)$, $\mathrm{VR}(q, S)$, $\mathrm{VR}(r, S)$, and so on.



Growing Circle

- Grow a circle from a point *x* on the plane



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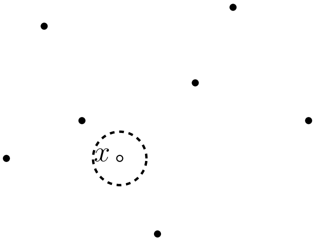

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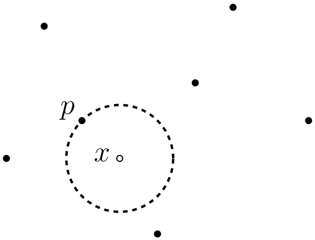

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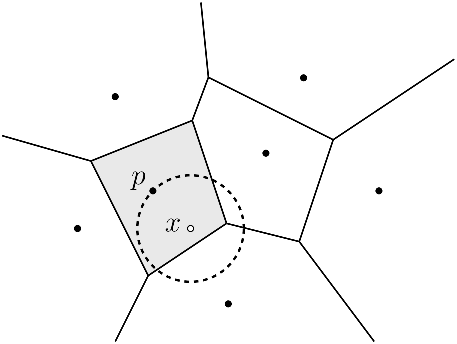

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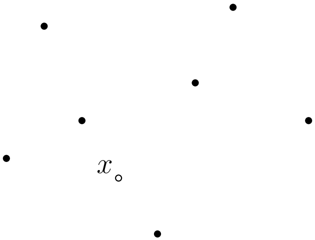

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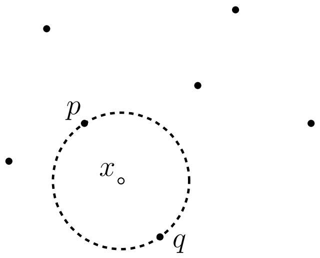

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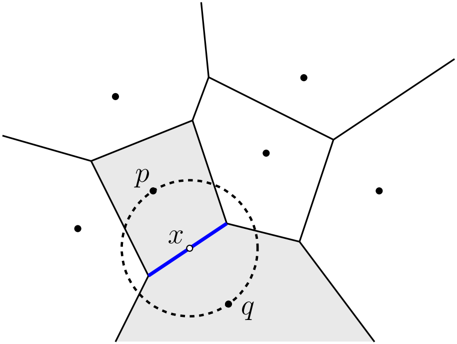

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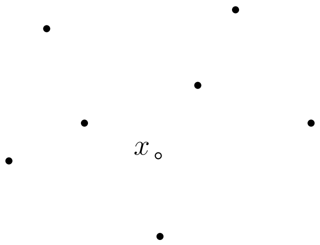

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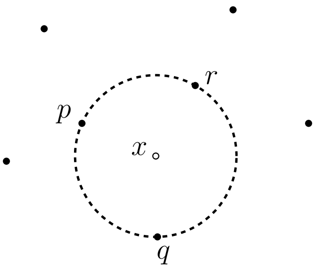

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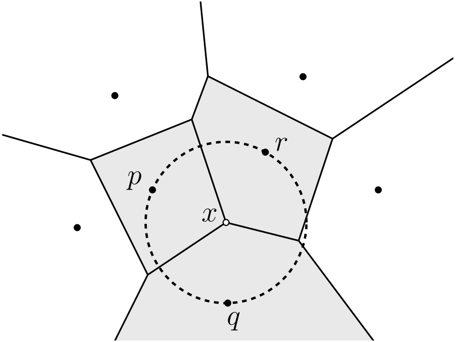
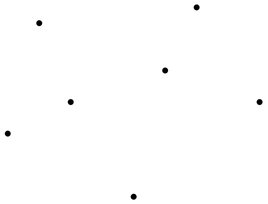


Wavefront Model (Growth Model)

- Grow circles from $\forall p \in S$ at unit speed



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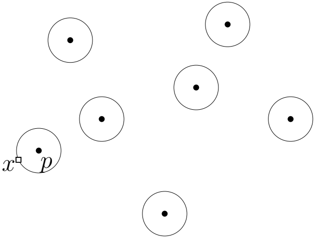

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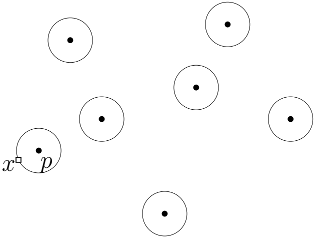

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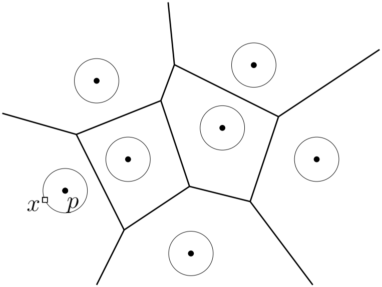

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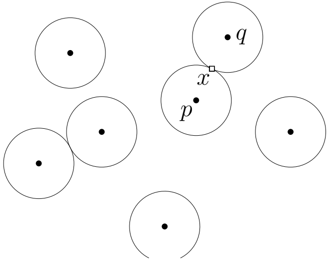

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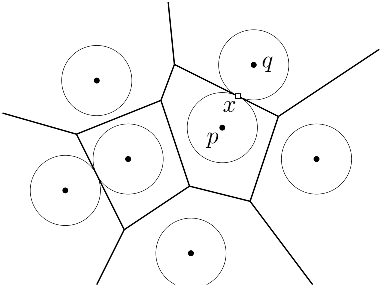

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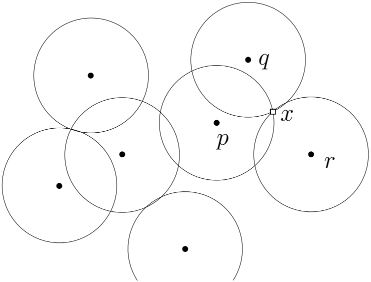

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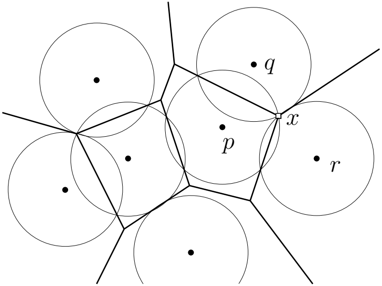
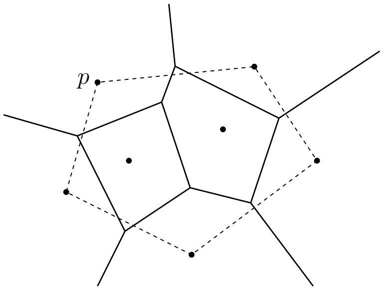


Unbounded Region

- VR(p, S) is unbounded if and only if p is a vertex of the convex hull of S.



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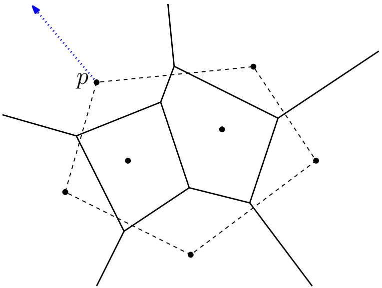

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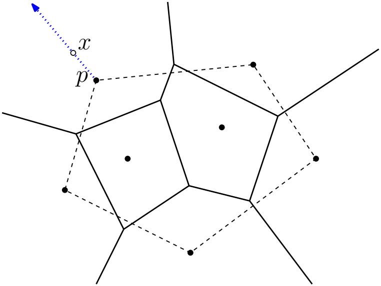

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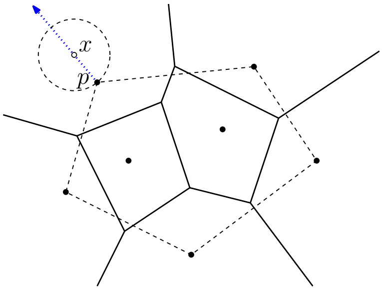

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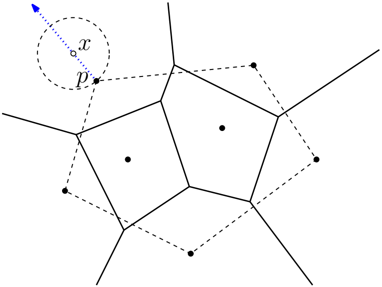
  - The bisector extends to the infinity.



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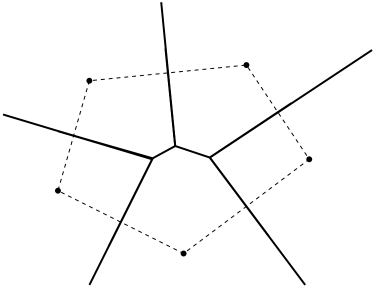
- If S is in convex position, $V(S)$ is a tree.



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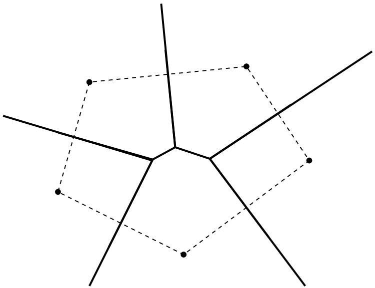
- If S is in convex position, $V(S)$ is a tree.
- An unbounded Voronoi edge corresponds to a hull edge.



Voronoi Diagram (Mathematic Definition)

- Voronoi Diagram $V(S)$

$$V(S) = R^2 \setminus (\bigcup_{p \in S} \mathrm{VR}(p, S)) = \bigcup_{p \in S} \partial \mathrm{VR}(p, S)$$

  - $\partial \mathrm{VR}(p, S)$ is the boundary of $\mathrm{VR}(p, S)$
    - $\partial \mathrm{VR}(p, S) \not\subset \mathrm{VR}(p, S)$
  - $V(S)$ is the union of all the Voronoi edges
- Voronoi Edge e between $\mathrm{VR}(p, S)$ and $\mathrm{VR}(q, S)$

$$e = \partial \mathrm{VR}(p, S) \cap \partial \mathrm{VR}(q, S)$$

- Voronoi Vertex v among $\mathrm{VR}(p, S)$, $\mathrm{VR}(q, S)$, and $\mathrm{VR}(r, S)$

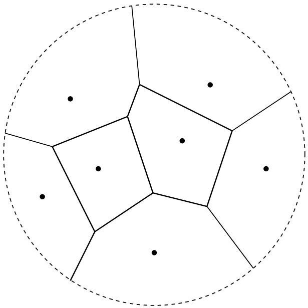
$$v = \partial \mathrm{VR}(p, S) \cap \partial \mathrm{VR}(q, S) \cap \partial \mathrm{VR}(r, S)$$

Complexity of $V(S)$

Theorem

$V(S)$ has $O(n)$ edges and vertices. The average number of edges of a Voronoi region is less than 6.

- Add a large curve Γ
  - Γ only passes through unbounded edges of $V(S)$
  - Cut unbounded pieces outside Γ
  - One additional face and several edges and vertices.



Complexity of $V(S)$

Theorem

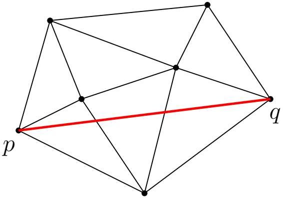
$V(S)$ has $O(n)$ edges and vertices. The average number of edges of a Voronoi region is less than 6.

- Euler's Polyhedron Formula: $v - e + f = 1 + c$
  - v: # of vertices, e: # of edges, f: # of faces, and c: # number of connected components.
- An edge has two endpoints, and a vertex is incident to at least three edges.
  - $3v \leq 2e \rightarrow v \leq 2e/3$
- $f = n + 1$ and $c = 1$
  - $v = 1 + c + e - f = e + 1 - n \leq 2e/3 \rightarrow e \leq 3n - 3$
  - $e = v + f - 1 - c = v + n - 1 \geq 3v/2 \rightarrow v \leq 2n - 2$
- Average number of edges of a region $\leq (6n - 6)/n < 6$

Triangulation

Definition

Given a set S of points on the plane, a triangulation is maximal collection of non-crossing line segments among S.



Crossing ($\overline{pq}$)

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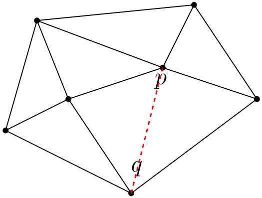

Not Maximal ($\overline{pq}$ is allowable)

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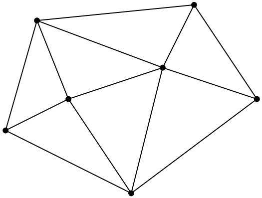
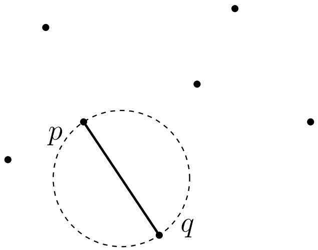


Triangulation

Delaunay Edge

Definition

An edge $\overline{pq}$ is called **Delaunay** if there exists a circle passing through p and q and containing no other point in its interior.



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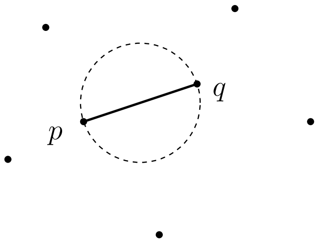

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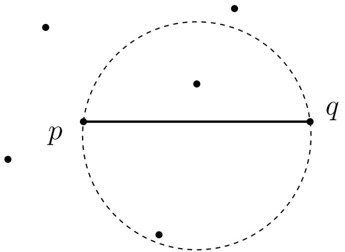
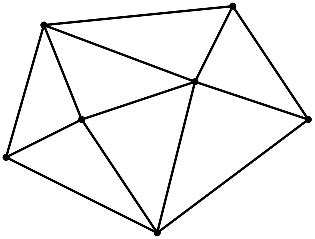


$\overline{pq}$ is **NOT** Delaunay

Delaunay Triangulation

Definition

A **Delaunay Triangulation** is a triangulation whose edges are all Delaunay.



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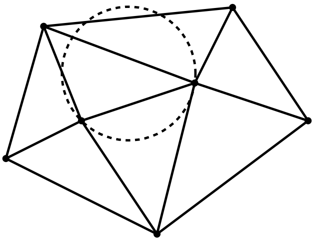

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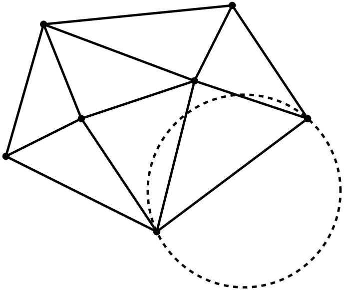

Delaunay Triangulation

Definition

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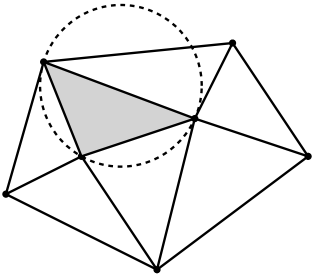
- For each face, there exists a circle passing all its vertices and containing no other point.



General Position Assumption

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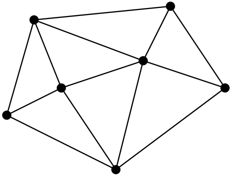
- There is a unique Delaunay triangulation.

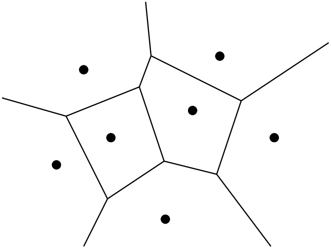
Duality

Theorem

Under the general position assumption, the Delaunay triangulation is a dual graph of the Voronoi diagram.

- A site $p \leftrightarrow$ a Voronoi region $VR(p, S)$





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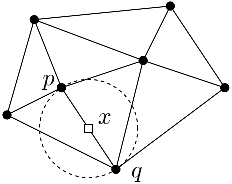

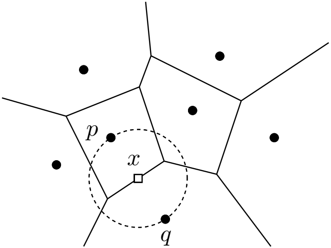

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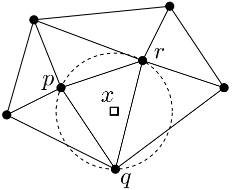
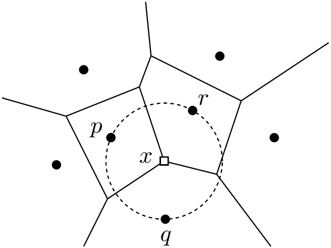
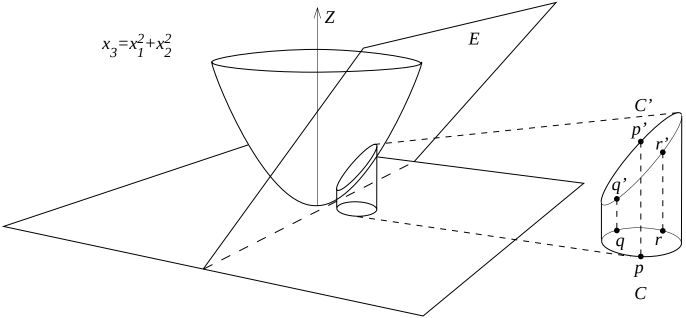
- A Delaunay triangle Δpqr $\leftrightarrow$ a Voronoi vertex among $VR(p, S)$, $VR(q, S)$ and $VR(r, S)$





Geometric Transformation from 2D to 3D

- A paraboloid $P = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = x_3\}$ in 3D
- For a point $x = (x_1, x_2)$ in 2D, $x' = (x_1, x_2, x_1^2 + x_2^2)$ is its lifted image in 3D
  - $x' \leftarrow$ vertical projection from x to P
- For a set A of points in 2D, its lifted image $A' = \{x' = (x_1, x_2, x_1^2 + x_2^2) \mid x = (x_1, x_2) \in A\}$



Circle in 2D $\leftrightarrow$ Planar Curve in P

Lemma

Let C be a circle in the plane. Then C' is a planar curve on the paraboloid P

- C is given by $r^2 = (x_1 - c_1)^2 + (x_2 - c_2)^2$
  - $r^2 = x_1^2 + x_2^2 - 2x_1c_1 - 2x_2c_2 + c_1^2 + c_2^2$
- C' satisfies $x_1^2 + x_2^2 = x_3$
- Substituting $x_1^2 + x_2^2$ by x_3, we obtain a plane E

$$x_3 - 2x_1c_1 - 2x_2c_2 + c_1^2 + c_2^2 - r^2 = 0$$

- $C' = P \cap E$
- Intersection between E and P is a planar curve

Lower Convex Hull

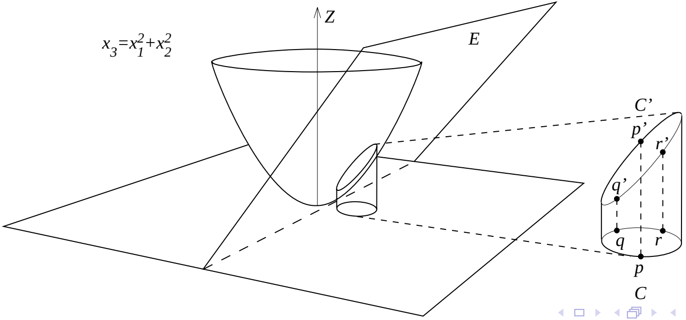
- S' on $P \rightarrow S'$ in convex position
- Each point of S' is a vertex of $conv(S')$
- Lower convex hull $lconv(S')$ of S' is the part of $conv(S')$ visible from $x_3 = -\infty$

Duality between $\mathrm{DT}(S)$ and $lconv(S')$ (1)

Theorem

The Delaunay triangulation $\mathrm{DT}(S)$ equals to the vertical projection onto the x_1x_2-plane of the lower convex hull of S'

- $p, q, r \in S$. C: circumcircle of p, q, r
- C' lies on a plane E defined by p', q', r'
- a point x inside $C \leftrightarrow$ lifted image x' below E



Duality between $\mathrm{DT}(S)$ and $lconv(S')$ (2)

Theorem

The Delaunay triangulation $\mathrm{DT}(S)$ equals to the vertical projection onto the x_1x_2-plane of the lower convex hull of S'

- p, q, r defines a triangle of $\mathrm{DT}(S)$
  $\leftrightarrow$ no point of S in C defined by p, q, r
  $\leftrightarrow$ no point of S' below E defined by p', q', r'
  $\leftrightarrow$ p', q', r' defines a facet of $lconv(S')$
- Computing a convex hull in 3D takes $O(n \log n)$ time
  - $V(S)$ in $O(n \log n)$ time

Another Viewpoint of paraboloid

- For each $s = (s_1, s_2) \in S$, a paraboloid

$$P_s = \{(x_1, x_2, x_3) \mid x_3 = (x_1 - s_1)^2 + (x_2 - s_2)^2\}$$

  - For each $x = (\sigma_1, \sigma_2)$ in $x_1 x_2$ plane, vertical distance from x to P_s is $d(x, s)^2$
- Opaque and of pairwise different colors
- Looking from $x_3 = -\infty$ upward $\rightarrow V(S)$
- Vertical from x upward first hits $P_s \rightarrow x \in \mathrm{VR}(p, S)$
- $P_s \cap P_t \rightarrow B(s, t)$
- Lower envelope of $\bigcup_{s \in S} P_s \rightarrow V(S)$

Wavefront model revisited

- $P_s = \{(x_1, x_2, x_3) | x_3 = f((x_1 - s_1)^2 + (x_2 - s_2)^2)\}$
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  - Cones of slope $45°$ with apices at sites $s \in S$
- Expanding circles C_s from sites $s \in S$ at equal unit speed
  - time t = radius r
  - $r^2 = (x_1 - s_1)^2 + (x_2 - s_2)^2$
  - x_3 = $\sqrt{(x_1 - s_1)^2 + (x_2 - s_2)^2}$ = radius = time

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  - $x_3 = \sqrt{(x_1 - s_1)^2 + (x_2 - s_2)^2}$ = radius = time
- x first hit by $C_s \leftrightarrow$ upward vertical projection from x first hit P_s

Post Office Problem

Post Office Problem

Given a query point $x = (x_1, x_2)$, answer the closest post office p among S

- $p \in S$ is the closest post office to x iff x belong to $\mathrm{VR}(p, S)$
  1. Compute $V(S)$ in $O(n \log n)$ time
  2. Construct a point location data structure for $V(S)$ in $O(n \log n)$ time
- Answer each query in $O(\log n)$ time after $O(n \log n)$-time preprocessing
  - Locating x in $\mathrm{VR}(p, S)$ takes $O(\log n)$ time

Nearest Neighbors

Nearest Neighbors

Given a set S of points, for each $s \in S$, compute its nearest neighbor among $S \setminus \{s\}$

Lemma

t is the nearest neighbor of s only if (s, t) is an edge of $\mathrm{DT}(S)$

- A circle growing from s will first hit t
  $\rightarrow$ There is a circle touching s and t but empty of $S \setminus \{s, t\}$

Theorem

The nearest neighbors can be computed in $O(n \log n)$ time

- $\mathrm{DT}(S)$ can be computed in $O(n \log n)$ time
- Since $|\mathrm{DT}(S)|$ is $O(n)$, computing the nearest neighbor from $\mathrm{DT}(S)$ takes $O(n)$ time

Largest Empty Circle

Largest Empty Circle

Given a set S of points inside a conex polygon A, find the largest circle C whose center is located in A and which enclose no point of S.

Lemma

C must touch two or three points of S

- touch no site: expand it until it touches one sites p
- touches one site p: expand it until it touches p and q
- touches p and q: expand it along $B(p, q)$ until
  1. the center of C hit the boundary of A
  2. C hit the third site r

Theorem

The center of C is either a Voronoi vertex of $V(S)$ or the intersection between a Voronoi edge and the boundary of A

Minimum Spanning Tree

Given a set S of points, compute a tree $\mathrm{MST}(S)$ connecting all sites of S with the minimum total length

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If S is partitioned into A and B, the shortest edge (a, b) satisfying $a \in A$ and $b \in B$ is an edge of $\mathrm{MST}(S)$

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Theorem

An edge of $\mathrm{MST}(S)$ is an edge of $\mathrm{DT}(S)$

- Each shortest edge (a, b), $a \in A$ and $b \in B$, is an edge of $\mathrm{DT}(S)$.
- The circle whose diameter $\overline{ab}$ enclose no other sites
  - If it contains a site $a' \in A$, $d(a', b) < d(a, b)$

Thank You!!

Divide and Conquer

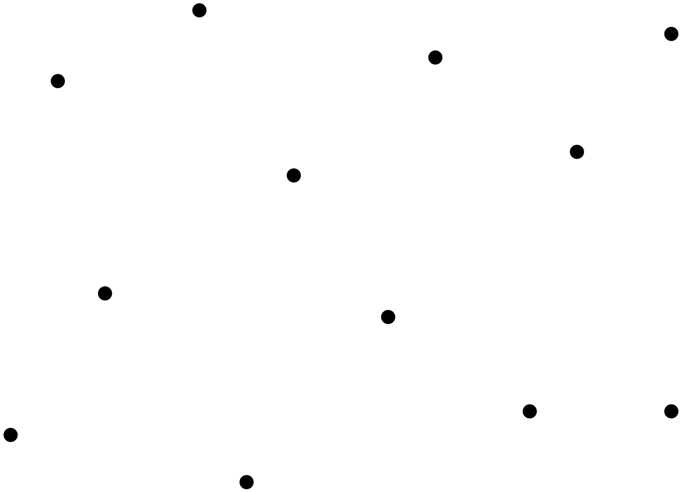
- Basic Steps
  1. Divide an instance into c equal-size sub-instances
     - If the instance is extremely small, solve it directly instead
  2. Recursively compute the sub-solution for each sub-instance
     - Until a sub-instance can be solved in $O(1)$ time
  3. Merge all the c sub-solutions into one solution
- If both Divide and Merge take linear time,

$$T(n) = cT(n/c) + O(n) => O(n \log n)$$

  - The i^{th} level has c^i parts and each part has n/c^i elements
  - A level takes $c^i \times O(n/c^i) = O(n)$ time
  - $O(\log n)$ levels

Example: Convex Hull

- $n = 12$ and $c = 2$



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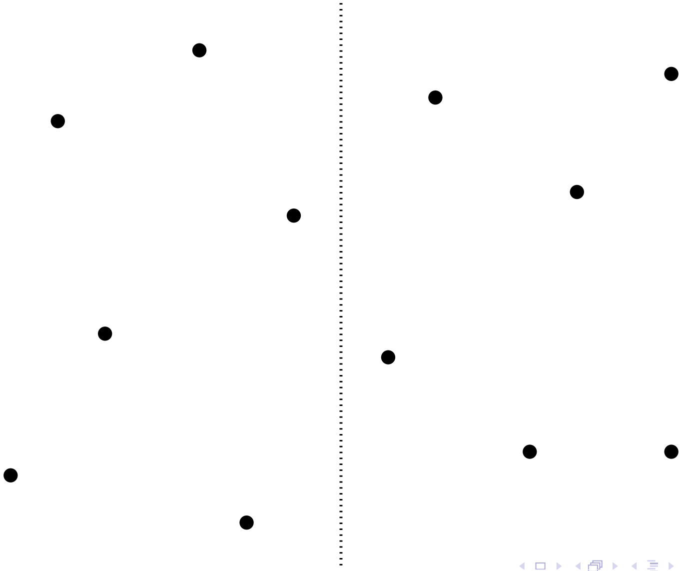

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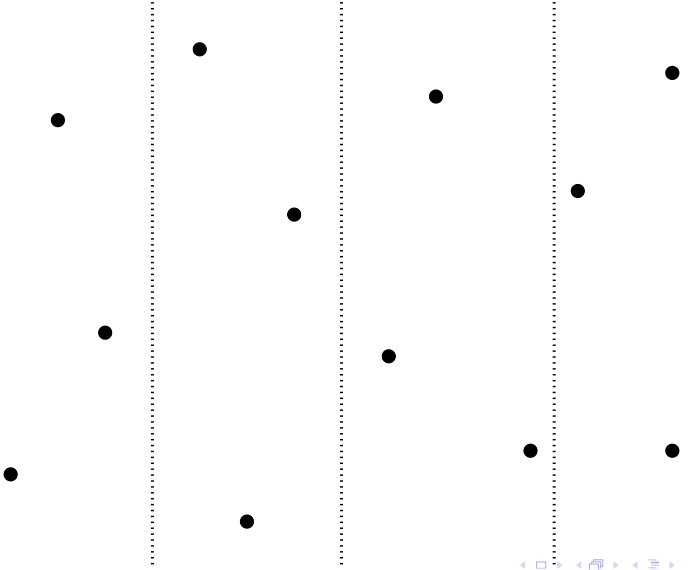

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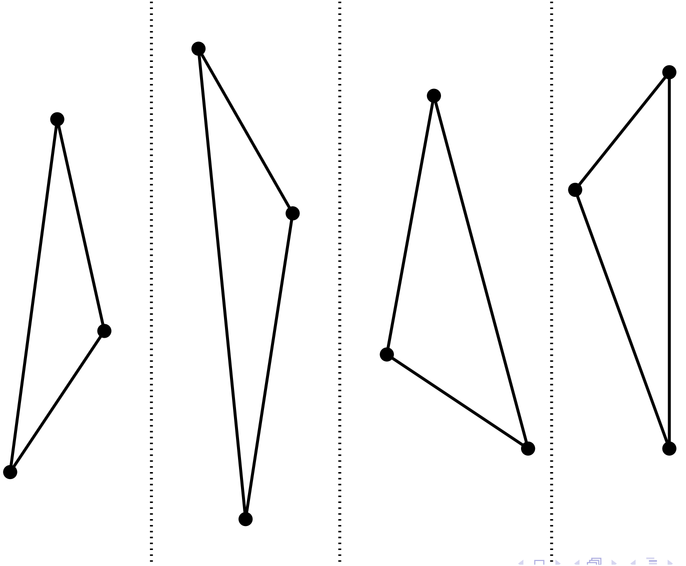

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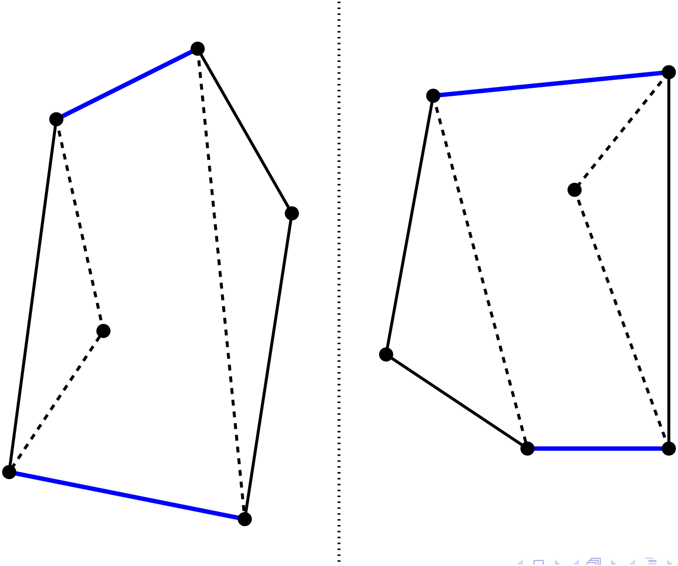

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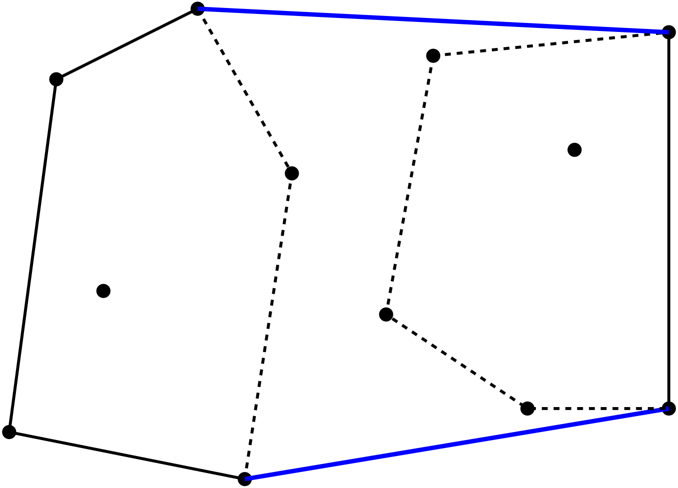

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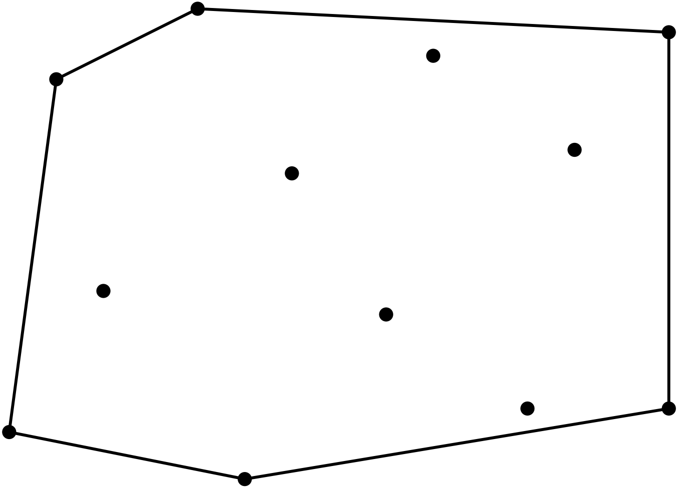
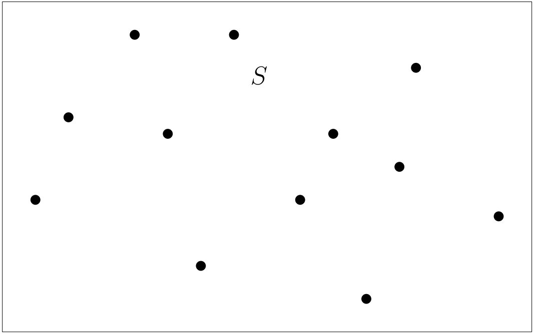


D & C for Voronoi Diagram

- Divide and Conquer to Compute $V(S)$
  1. Use a vertical line to partition S into L and R where $|L| \sim |R|$
     - If $|S|$ is a constant, directly compute $V(S)$ instead
  2. Recursively compute $V(L)$ and $V(R)$
  3. Merge $V(L)$ and $V(R)$ into $V(S)$



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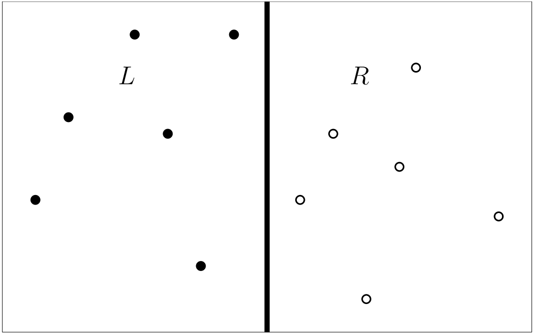

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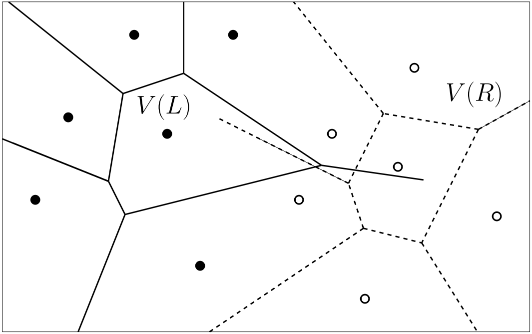

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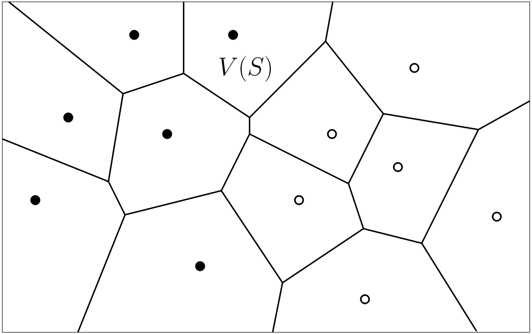
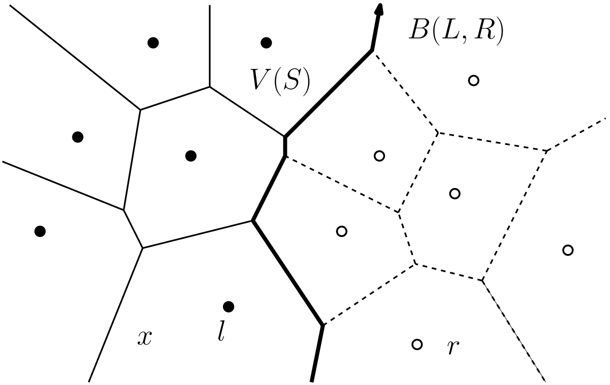


Merge Chain

- **Merge Chain** $B(L, R)$ consists of Voronoi edges between $V(l, S)$ and $V(r, S)$ where $l \in L$ and $r \in R$.



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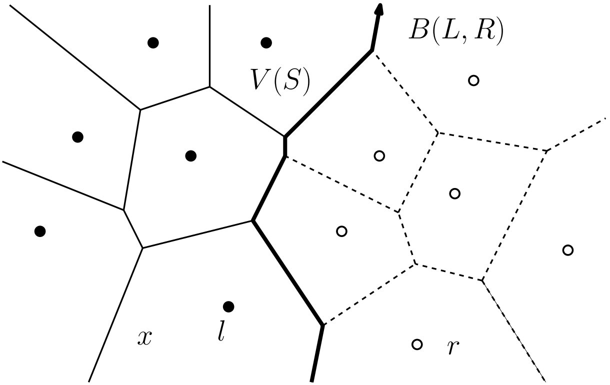

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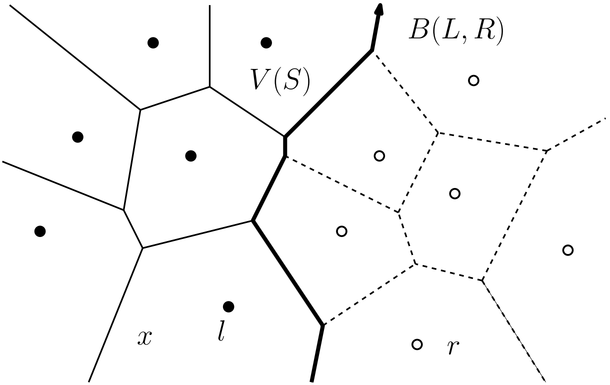

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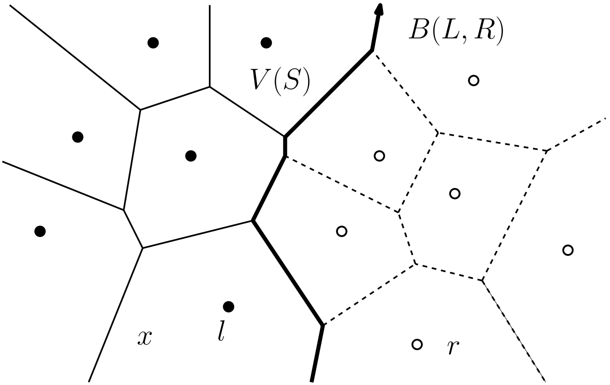
  - $B(L, R) = \{x \in R^2 \mid d(x, L) = d(x, R)\}$



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  - Voronoi edges between L's regions and R's regions
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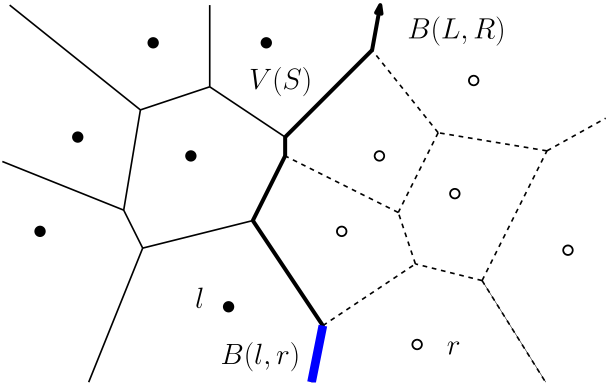
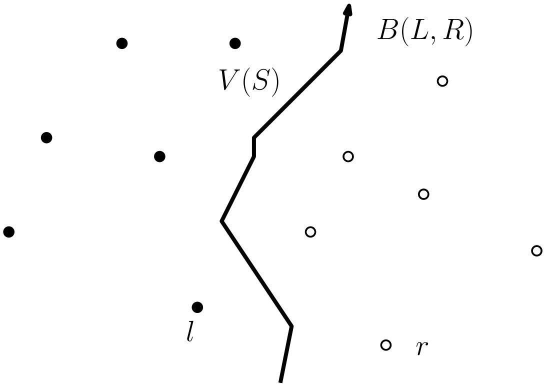
  - $B(L, R) = \{x \in R^2 \mid d(x, L) = d(x, R)\}$
    - $\forall e \in B(L, R)$ belongs to $B(l, r)$, $l \in L$ and $r \in R$



If We Have the Merge Chain...

- Compute $V(S)$
  - Remove the part of $V(L)$ right to $B(L, R)$
  - Remove the part of $V(R)$ left to $B(L, R)$
  - Glue the two remaining parts



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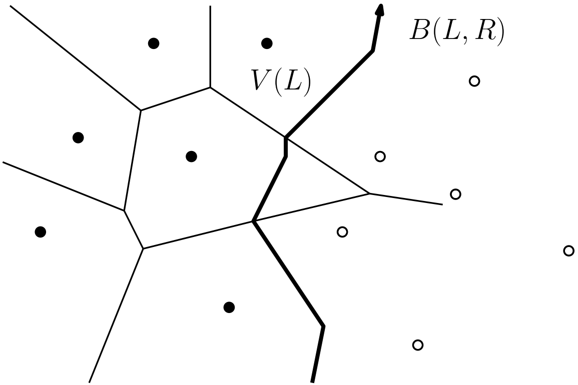

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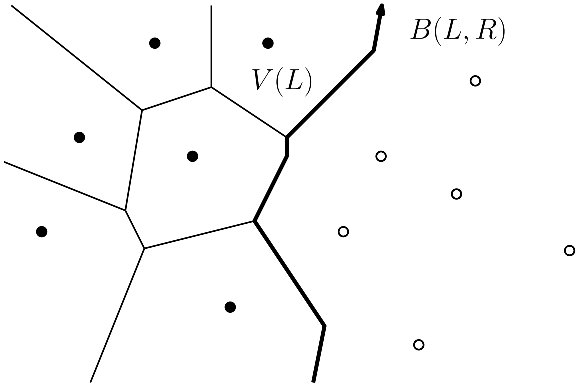

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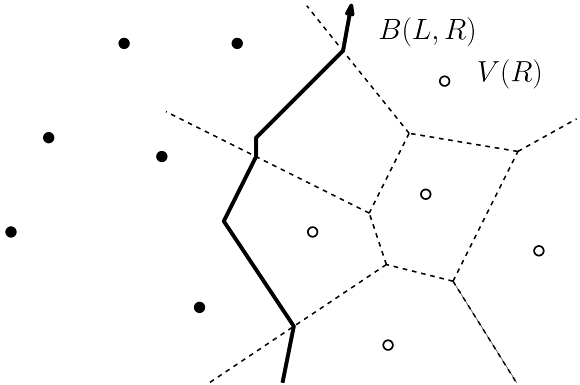

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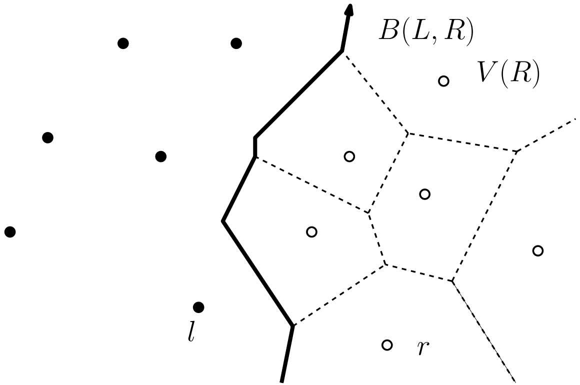

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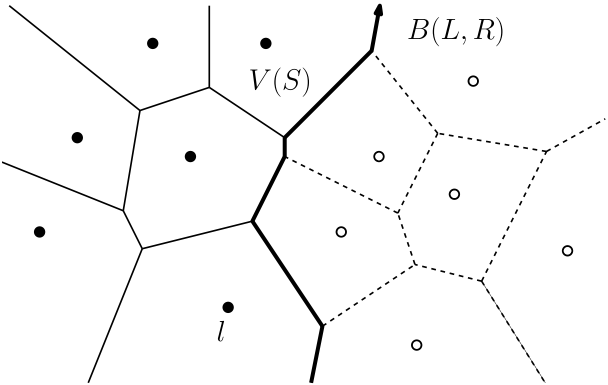
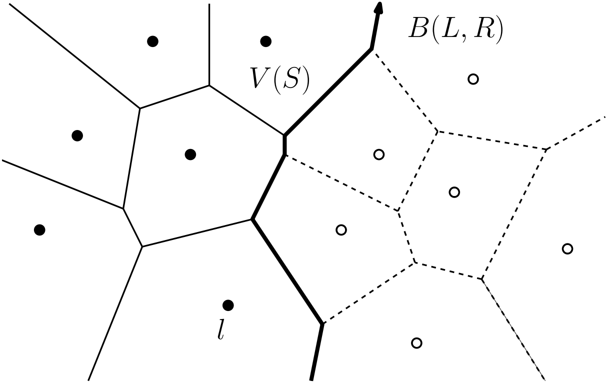


If We Have the Merge Chain...

- Compute $V(S)$
  - Remove the part of $V(L)$ right to $B(L,R)$
  - Remove the part of $V(R)$ left to $B(L,R)$
  - Glue the two remaining parts
- $B(L,R)$ is a part of $V(S)$



$B(L, R)$ is y-monotone

Lemma

$B(L, R)$ is y-monotone

- Let b be an edge of $B(L, R)$.
- b belongs to $V(l, S)$ and $V(r, S)$, $l \in L$ and $r \in R$
- Let b be directed such that l in the left of b
- x-coordinate of $l <$ x-coordinate of r
  $\rightarrow$ b must be **upward**

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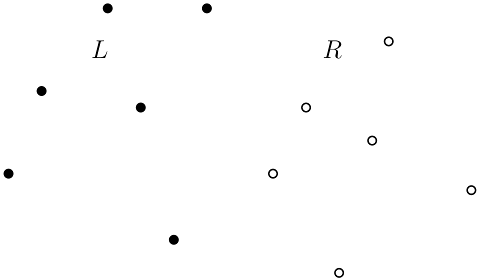
Computing $B(L, R)$

- Find the bottom edge e of $B(L, R)$ as a starting edge
  - e is unbounded and extends to $-\infty$
- Trace out $B(L, R)$ from e

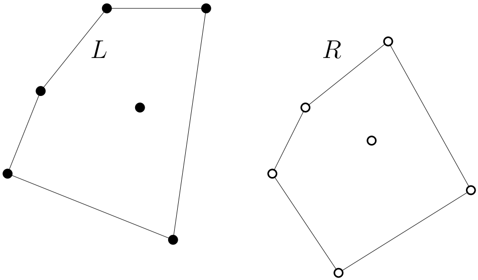
Finding a starting edge of $B(L, R)$

- An unbounded Voronoi edge corresponds to an edge of the convex hull $conv(S)$ of S



Finding a starting edge of $B(L, R)$

- An unbounded Voronoi edge corresponds to an edge of the convex hull $conv(S)$ of S
  - Build $conv(L)$ and $conv(R)$
    - $V(L)$ and $V(R)$ are known.



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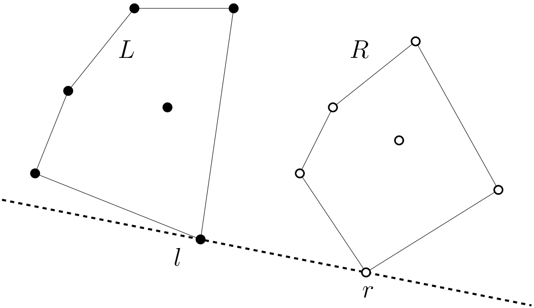
  - Compute a tangent edge $\overline{lr}$ between $conv(L)$ and $conv(R)$



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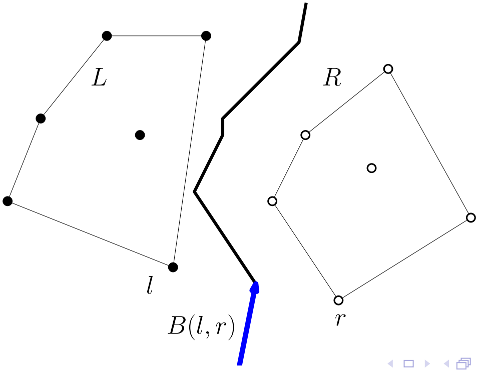
  - $B(l, r)$ is the starting edge



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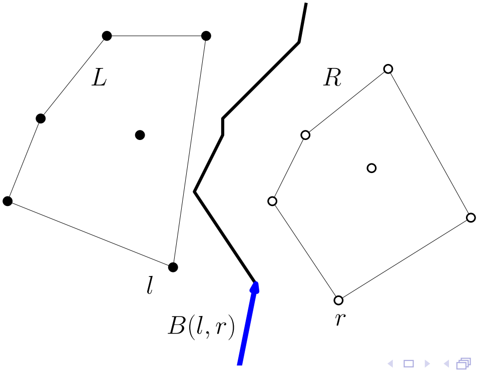
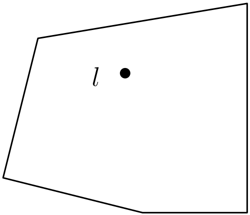
  - Compute a tangent edge $\overline{lr}$ between $conv(L)$ and $conv(R)$
  - $B(l, r)$ is the starting edge
- $O(|L| + |R|)$ time



Trace $B(L, R)$ in $V(l, L)$ (1)

- $B(L, R) \cap V(l, L)$
  - $B(L, R)$ enters $V(l, L)$ at v along $B(l, r_1)$



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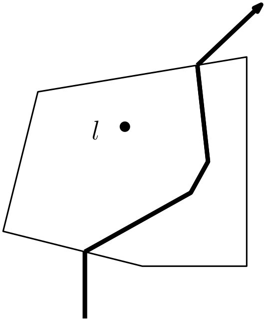

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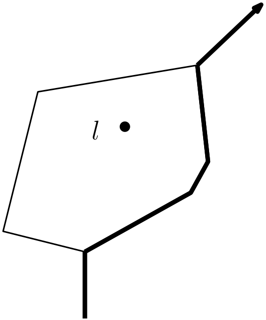

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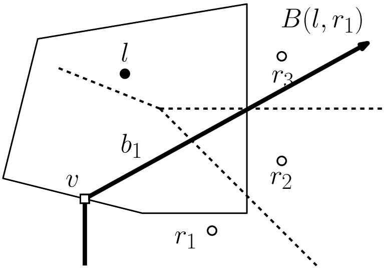

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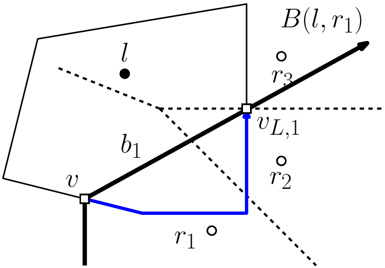
  - Counterclockwise along $\partial V(l, L)$ from v to find $v_{L,1} \in B(l, r_1)$
    - After $v_{L,1}$, $V(l, L) \to V(l', L)$, and $B(l, r_1) \to B(l', r_1)$



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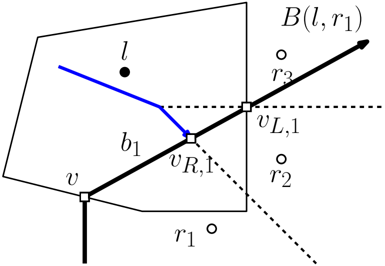
    - After $v_{L,1}$, $V(l, L) \to V(l', L)$, and $B(l, r_1) \to B(l', r_1)$
  - Clockwise along $\partial V(r_1, R)$ to find $v_{R,1} \in B(l, r_1)$
    - After $v_{R,1}$, $V(r_1, R) \to V(r_2, R)$, and $B(l, r_1) \to B(l, r_2)$



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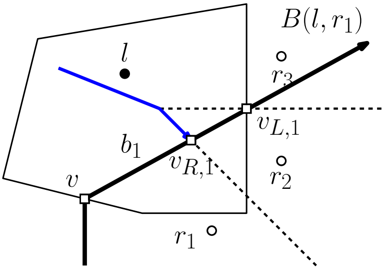
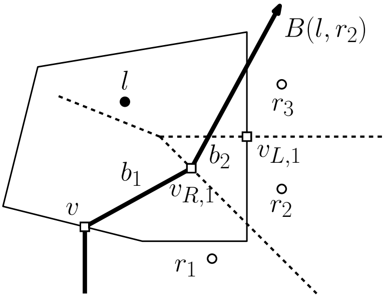
    - After $v_{R,1}$, $V(r_1, R) \to V(r_2, R)$, and $B(l, r_1) \to B(l, r_2)$
  - $v_{R,1}$ earlier than $v_{L,1}$, and $B(l, r_1) \to B(l, r_2)$ at $v_{R,1}$



Trace $B(L, R)$ in $V(l, L)$ (2)

- $B(L, R) \cap V(l, L)$
  - $B(L, R)$ from $v_{R,1}$ along $B(l, r_2)$



Trace $B(L,R)$ in $V(l,L)$ (2)

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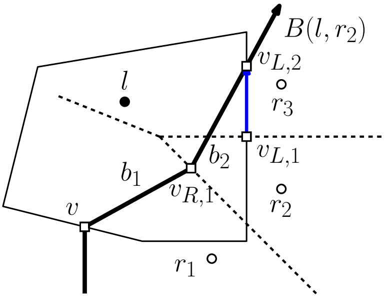
  - $B(L,R)$ from $v_{R,1}$ along $B(l,r_2)$
  - Counterclockwise along $\partial V(l,L)$ from $v_{L,1}$ to find $v_{L,2} \in B(l,r_2)$
    - After $v_{L,2}$, $V(l,L) \to V(l',L)$, and $B(l,r_2) \to B(l',r_2)$



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  - Counterclockwise along $\partial V(l,L)$ from $v_{L,1}$ to find $v_{L,2} \in B(l,r_2)$
    - After $v_{L,2}$, $V(l,L) \rightarrow V(l',L)$, and $B(l,r_2) \rightarrow B(l',r_2)$
  - Clockwise along $\partial V(r_2,R)$ from $v_{R,1}$ to find $v_{R,2} \in B(l,r_2)$
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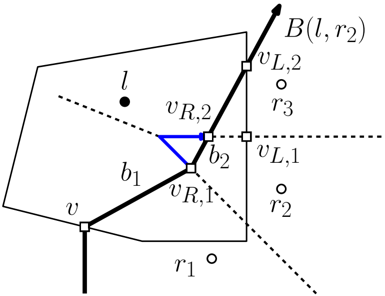

Trace $B(L, R)$ in $V(l, L)$ (2)

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  - $B(L, R)$ from $v_{R,1}$ along $B(l, r_2)$
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    - After $v_{L,2}$, $V(l, L) \to V(l', L)$, and $B(l, r_2) \to B(l', r_2)$
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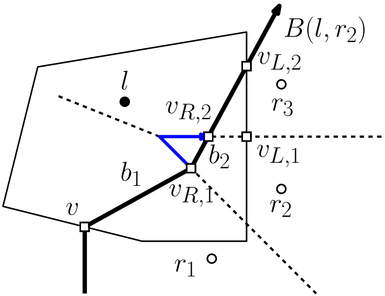

Trace $B(L, R)$ in $V(l, L)$ (3)

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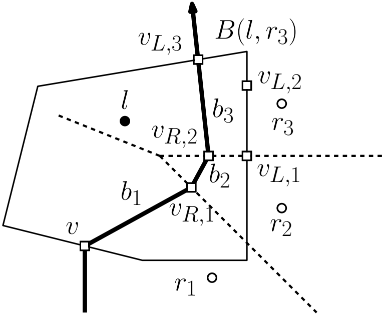
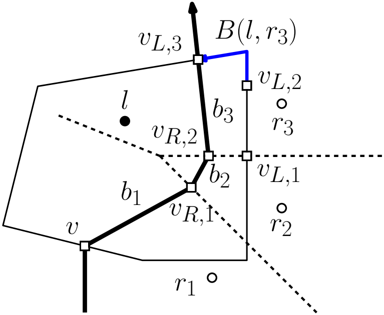
  - $B(L, R)$ from $v_{R,2}$ along $B(l, r_3)$



Trace $B(L, R)$ in $V(l, L)$ (3)

- $B(L, R) \cap V(l, L)$
  - $B(L, R)$ from $v_{R,2}$ along $B(l, r_3)$
  - Counterclockwise along $\partial V(l, L)$ from $v_{L,2}$ to find $v_{L,3} \in B(l, r_3)$
    - After $v_{L,3}$, $V(l, L) \to V(l'', L)$, and $B(l, r_3) \to B(l'', r_3)$



Trace $B(L, R)$ in $V(l, L)$ (3)

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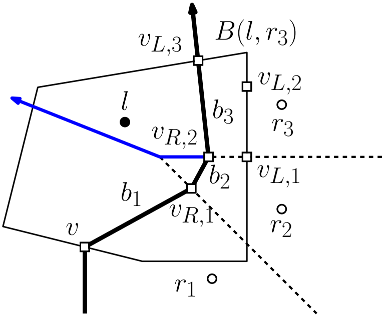
  - Counterclockwise along $\partial V(l, L)$ from $v_{L,2}$ to find $v_{L,3} \in B(l, r_3)$
    - After $v_{L,3}$, $V(l, L) \to V(l'', L)$, and $B(l, r_3) \to B(l'', r_3)$
  - Clockwise along $\partial V(r_3, R)$ from $v_{R,2}$ to find $v_{R,3} \in B(l, r_3)$
    - No $v_{R,3}$



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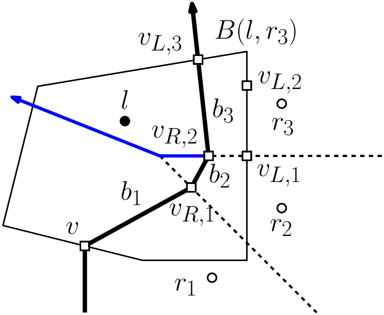

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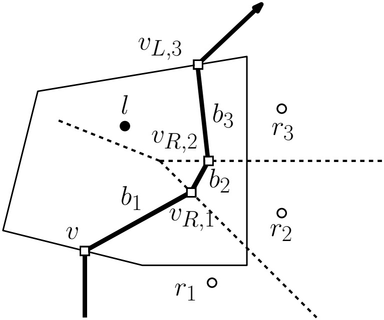

Time to Compute $B(L, R)$

Lemma

Computing $B(L, R)$ takes $O(|L| + |R|) = O(|S|)$ time

- Finding the starting edge takes $O(|L| + |R|)$ time
- Traversing $V(L)$ takes $O(|V(L)|)$ time ($V(R) \rightarrow O(|V(R)|)$)
  - Enter each $V(l, L)$ and its boundary at most once
- There are $O(|B(L, R)|)$ intersections
  - Each edge of $B(L, R)$ makes two intersections.
- $O(|L| + |R|) + O(|V(L)|) + O(|V(R)|) + O(|B(L, R)|)$ $= O(|L| + |R| + |L| + |L| + |S|) = O(|S|)$
  - $O(|V(L)|) = O(|L|)$, $O(|V(R)|) = O(|R|)$, and $O(|B(L, R)|) = O(|S|)$

Time Complexity

Theorem

The divide-and-conquer algorithm computes $V(S)$ in $O(n \log n)$ time

- Sorting takes $O(n \log n)$ time (only do once)
- A sub-instance $S' \subset S$ takes $O(|S'|)$ time
  - Partitioning S' into L' and R' takes $O(|S'|)$ time
  - Computing $B(L', R')$ takes $O(|S'|)$ time
- The i^{th}-level takes $O(2^i) \times O(n/2^i) = O(n)$ time
  - $O(2^i)$ sub-instances each with $O(n/2^i)$ sites
- $O(\log n)$ levels

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- $O(\log n)$ levels
- $T(n) = O(n) + 2 \cdot T(n/2) => T(n) = O(n \log n)$