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Theoretical Aspects of Intruder Search

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The manuscript will be successively extended during the lecture in the Wintersemester. Hints and comments for improvements can be given to Elmar Langetepe by E-Mail `elmar.langetepe@informatik.uni-bonn.de`. Thanks in advance!

Lemma 21 *Any contiguous monotone strategy for T' can be translated to a contiguous monotone strategy for T with the same number k of agents.*

Proof. Let $e' = (x, y)$ and $e'' = (y, z)$ be links stemming from the extension of a link e. If q guards move from x to y or z to y, they stay in there place in T. If q guards move from y to x or from y to z, they will move from z to x or from x to z in T, respectively. □

The other way round, any strategy for T is also a strategy for T'.

Lemma 22 *Any contiguous monotone strategy for T with k agents can be translated to a contiguous monotone strategy for T' with the same number k of agents.*

Proof. A move along an edge e in T is splitted into two moves along e' and e'' in T'. If the move clears e, then $q \geq w(e)$ have traversed e. From the construction q searchers are also enough for $w(e) = w(e') = w(e'')$ and the weight $w(e)$ of the intermediate vertex. □

We collect our results:

Proof of Theorem 17: From Lemma 21 we conclude $\mathrm{cs}(T') \leq \mathrm{cs}(T)$. From Lemma 18 we obtain a connected crusade of frontier $\leq \mathrm{cs}(T)$ in T'. From Lemma 19 we conclude that there is a progressive connected crusade of frontier $\leq \mathrm{cs}(T)$ in T'. From Lemma 20 we obtain a monotone contiguous search strategy using $\leq \mathrm{cs}(T)$ guards in T' and we can assume that all searchers are initially at a single starting vertex v_1. From Lemma 22 we conclude that there is also an optimal monotone contiguous search strategy that starts with all guards in a single vertex.

2.2.5 Designing a monotone strategy for unit weights

By Theorem 17 we can start strategy from a single vertex v and we can consider monotone strategies. Therefore, we design an optimal strategy for any starting vertex v and for the rooted tree T_v we compute the minimum number, $\mathrm{cs}(T_v)$, of agents required for starting in v. Finally we have $\mathrm{cs}(T) = \min_{v \in T} \mathrm{cs}(T_v)$.

An optimal monotone strategy for computing, $\mathrm{cs}(T_v)$, will also give an ordering all vertices z of T_v, stating which subtree, say $T_v(z)$, of T_v w.r.t. root v is fully cleared first. For this we can also consider the subtree $T_v(z)$ alone with root z and ask for $\mathrm{cs}(T_v(z))$ for short and an optimal monotone strategy.

We denote the children of the vertex z of the subtree $T_v(z)$ of T_v by $z_1, \ldots, z_d$ w.r.t. the order $\mathrm{cs}(T_v(z_i)) \geq \mathrm{cs}(T_v(z_{i+1}))$ for $i = 1, \ldots, d-1$. An example is given in Figure 2.11. Now, we can prove the main structural result. Unfortunately, there is a flaw in the proof of Barrière at al. and we can only proof the statement for unit weighted trees. The flaw is precisely marked in the proof below.

Lemma 23 *Let $z_1, \ldots, z_d$ be the $d \geq 2$ children of a vertex z in T_v and assume that $cs(T_v(z_i)) \geq cs(T_v(z_{i+1}))$ for $i = 1, \ldots, d-1$. We have*

$$cs(T_v(z)) = \max\{cs(T_v(z_1)), cs(T_v(z_2)) + w(z)\} \tag{2.5}$$

it the tree T is a tree with unit weights.

Proof. We can assume that $\mathrm{cs}(T_v(z)) \geq \mathrm{cs}(T_v(z_1))$ holds because we have to clear $T_v(z_1)$ before clearing $T_v(z)$. If in Equation 2.5 $\mathrm{cs}(T_v(z_1) \geq \mathrm{cs}(T_v(z_2) + w(z)$ holds, we can clear $T_v(z)$ by setting $w(z)$ on z and clear all $T_v(z_i)$ by $\mathrm{cs}(T_v(z_1)$ agents but $T_v(z_1)$ last. Note that also $w((z, z_i)) \leq w(z_i) \leq \mathrm{cs}(T_v(z_i)$ for all i for moving back from subtrees to z. Altogether, $\mathrm{cs}(T_v(z_1)$ agents are required and they are sufficient.

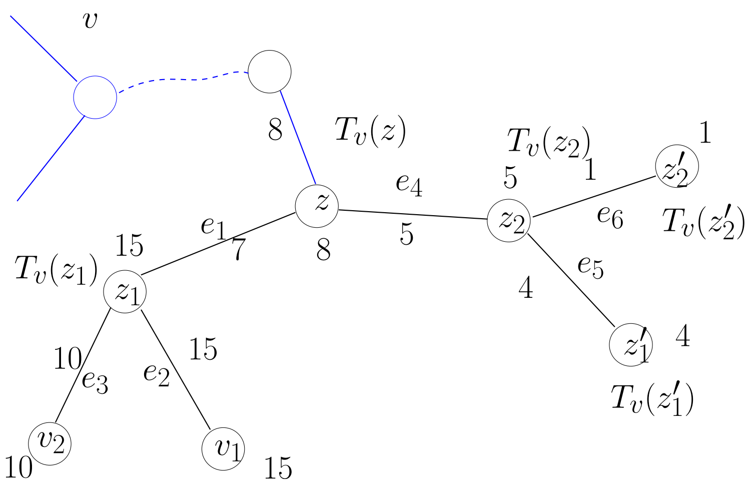


Figure 2.11: The rooted tree T_v has two subtrees $T_v(z_1)$ and $T_v(z_2)$ at vertex z. We have $\mathrm{cs}(T_v(z_1)) = 25$ and $\mathrm{cs}(T_v(z_2) = 6$ and $\mathrm{cs}(T_v(z_2)) + w(z) = 14 < 25 = \mathrm{cs}(T_v(z_1))$ which means that $\mathrm{cs}(T_v(z)) = 25$ holds. We leave $w(z)$ agents at z and clean $T_v(z_2)$ first. In $T_v(z_2)$ the same situation occurs, here $\mathrm{cs}(T_v(z'_1)) = 4$ and $\mathrm{cs}(T_v(z_2)) = 1$ but $\mathrm{cs}(T_v(z'_2)) + w(z_2) = 6 > 4 = \mathrm{cs}(T_v(z'_1)$. Therefore we require $\mathrm{cs}(T_v(z'_2)) + w(z_2) = 6$ agents, first we clean $T_v(z'_2)$ by 1 agent and block z_2 by 5 agents. Then we clean $T_v(z'_1)$ by 6 agents.

So let us assume that in Equation 2.5 $\mathrm{cs}(T_v(z_1)) < \mathrm{cs}(T_v(z_2)) + w(z)$ holds. We would like to prove that $\mathrm{cs}(T_v(z_2)) + w(z) - 1$ agents are not sufficient. We consider two cases:

1. **$T_v(z_2)$ is cleared before $T_v(z_1)$:** While $\mathrm{cs}(T_v(z_2))$ agents clear $T_v(z_2)$ there are only $w(z) - 1 = 0$ agents left for blocking a vertex in $T_v(z_1)$. Recontamination!

2. **$T_v(z_1)$ is cleared before $T_v(z_2)$):** While $\mathrm{cs}(T_v(z_1))$ agents clear $T_v(z_1)$ there are no more than $w(z) - 1 = 0$ agents left for blocking a vertex in $T_v(z_2)$ (because $\mathrm{cs}(T_v(z_1)) = \mathrm{cs}(T_v(z_2)))$. Recontamination!

The above statement do not hold for general weighted trees, because the fact that one only partially decontaminates $T_v(z_2)$ or $T_v(z_1)$ is not taken into account. For example, consider the vertex, say v with weight 5 in the center of Figure 2.12. and let z_1, z_2, and z_3 be the children of v below v from right to left. We have $\max\{\mathrm{cs}(T_x(z_1)), \mathrm{cs}(T_x(z_2)) + w(z)\} = \max\{8, 7 + 5\} = 12$ but we can recontaminate the subtree by 10 agents only, if we first clean z_3, leaving 5 agent at v. Then only clean vertex z_2 with one agent and leave this agent there. Then we clean $T_x(z_1)$ with the remaining 9 agents, and finally return to z_3 for the last part.

So $\mathrm{cs}(T_v(z_2)) + w(z)$ are required and are also sufficient by occupying z with $w(z)$ and clearing all $T_v(z_i)$ by $\mathrm{cs}(T_v(z_2))$ agents but $T_v(z_1)$ last with $\mathrm{cs}(T_v(z_2)) + w(z)$ agents. Note that also $w((z, z_i)) \leq w(z_i) \leq \mathrm{cs}(T_v(z_i))$ for all i for moving back from subtrees to z. □

The consequence of the above Lemma is, that we can compute $\mathrm{cs}(T_v)$ in $O(n)$ time by recursively applying Equation 2.5. Alternatively, we can start from the vertices.

Exercise 13 *Compute $cs(T_{v_4})$ for the tree in Figure 2.5 by the above recursive process.*

Corollary 24 *For a unit weighted tree T of size n and for a given starting vertex v we can compute the optimal monotone contiguous strategy starting at v in $O(n)$ time. An overall optimal contiguous strategy can be computed in $O(n^2)$.*

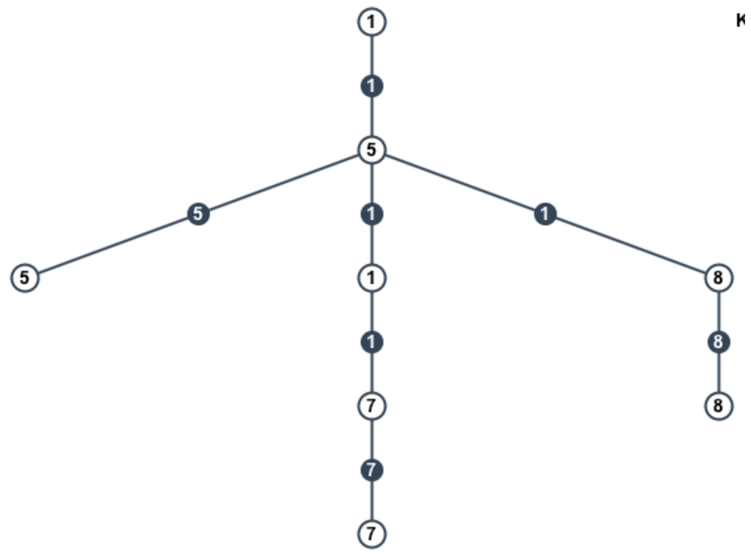


Figure 2.12: The flaw in the prove of Barriére et al. The recursion $\text{cs}(T_v(z)) = \max\{\text{cs}(T_v(z_1)), \text{cs}(T_v(z_2)) + w(z)\}$ does not hold for arbitrary weighted trees.

2.2.6 Computing an optimal contiguous Intruder Search Strategy for unit weights

We consider a message based algorithm that compute the optimal number of agents required for any starting vertex v.

The following local recursive labeling $\lambda_x(e)$ for the links $e = (x, y)$ adjacent to x will be sufficient. Let $e = (x, y)$ be a link incident to x.

1. If y is a leaf, set $\lambda_x(e) = w(y)$.
2. Otherwise, let d be the degree of y and let $x_1, \ldots, x_{d-1}$ be the incident vertices of y different form x. Let $\lambda_y(y, x_i) =: l_i$ and $l_i \geq l_{i+1}$. Then,

$$\lambda_x(e) := \max\{l_1, l_2 + w(y)\}\,.$$

For any link $e = (x, y)$ we will have two labels $\lambda_x(e)$ and $\lambda_y(e)$. By a *messages sending* technique, we can compute the labels $\lambda_x(e)$ and $\lambda_y(e)$ for alle edges $e = (x, y)$ in overall linear time. Note that we interpret any link $e = (x, y)$ as undirected, which means that $(x, y) = (y, x) = e$, more formally we could have used a notion $e = \{x, y\}$.

The message sending algorithm works as follows:

1. Start with the leaves and for any leaf y and for $e = (x, y)$ send a message $l = w(y)$ to x. After receiving this messages, x sets $\lambda_x(e) = l$.
2. Consider a vertex y of degree d that has received at least $d-1$ messages l_i from the incident certices $x_1, \ldots, x_{d-1}$ and let x be the remaining incident vertex. Let $l_i \geq l_{i+1}$. Send a message $l = \max\{l_1, l_2 + w(y)\}$ to x, after receiving the message x, set $\lambda_x((x, y)) = l$.

The above process can be applied sequentially, starting from the leaves. The process will not stop until we have send a message from x to y and y to x along any edge $e = (x, y)$. The process ends and in total $O(n)$ messages have been transmitted. An example is given in Figure 2.13. Keep in mind that we set $\lambda_x(e)$ meaning that x has received a message from e.

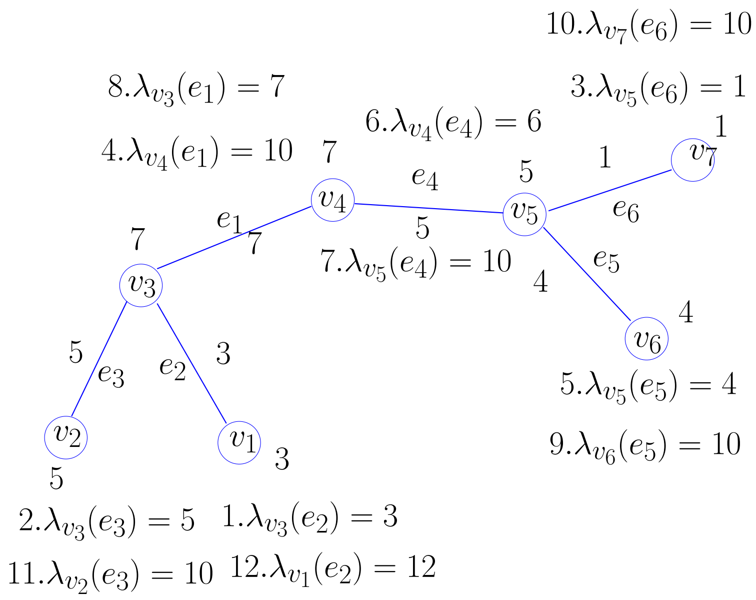


Figure 2.13: The message sending algorithm can easily work sequentially.

Lemma 25 *The links of a tree T can be labeled with labels λ_x by the above message sending algorithm by $O(n)$ messages in total.*

Finally, we would like to prove that for an edge $e = (x, y)$ the labeling algorithm indeed computes $\text{cs}(T_x(y))$ for the rooted tree T_x and its direct neighbor y. Note, that we can only proof the result for unit weighted trees.

Lemma 26 *For a unit weighted tree $T = (V, E)$ and an edge $e = (x, y) \in E$ we have $cs(T_x(y)) = \lambda_x(e)$.*

Proof. The proof goes by induction on the height $h(y)$ of $T_x(y)$. If y is a leaf we have $\lambda_x(e) = w(y)$ for $h(y) = 0$. The statement holds.

Assume that the statement holds for $0 \leq h(y) < k$ and consider $h(y) = k$. For edge $e = (x, y)$ let $x_1, \ldots, x_d$ be the $d \geq 1$ be the children of y in $T_x(y)$ and assume that $\lambda_y((y, x_i)) \geq \lambda_y((y, x_{i+1}))$ holds for $i = 1, \ldots, d-1$. We also have $T_y(x_i) = \lambda_y((y, x_i)$ by induction hypothesis and $T_y(x_i) = T_x(x_i)$ by definition. Therefore we also have $\text{cs}(T_x(x_i)) \geq \text{cs}(T_x(x_{i+1}))$ for $i = 1, \ldots, d-1$.

In Lemma 23 the recursion Equation 2.5 for $T_x(y)$ is exactly the same as step 2. $\lambda_x((x, y))$ for in the labeling process 2.2.6.

Therefore, we conclude $\text{cs}(T_x(y)) = \lambda_x(y)$. □

Finally, we have to compute the optimal number of agents and also a corresponding strategy. The first part can done as follows. We compute the minimum number of agents, $\mu(v)$ required for starting at a vertex v in the tree T.

For this we order all $\lambda_v((v, x_i)$ for all $i = 1, \ldots, d$ incident edges (v, x_i) so that $\lambda_v((v, x_i)) \geq \lambda_v((v, x_{i+1}))$ and compute

$$\mu(v) = \max\{\lambda_v((v, x_1)), \lambda_v((v, x_2)) + w(v)\} \,. \tag{2.6}$$

See for example the computation of $\mu(v_3)$ and $\mu(v_5)$ in Figure 2.14.

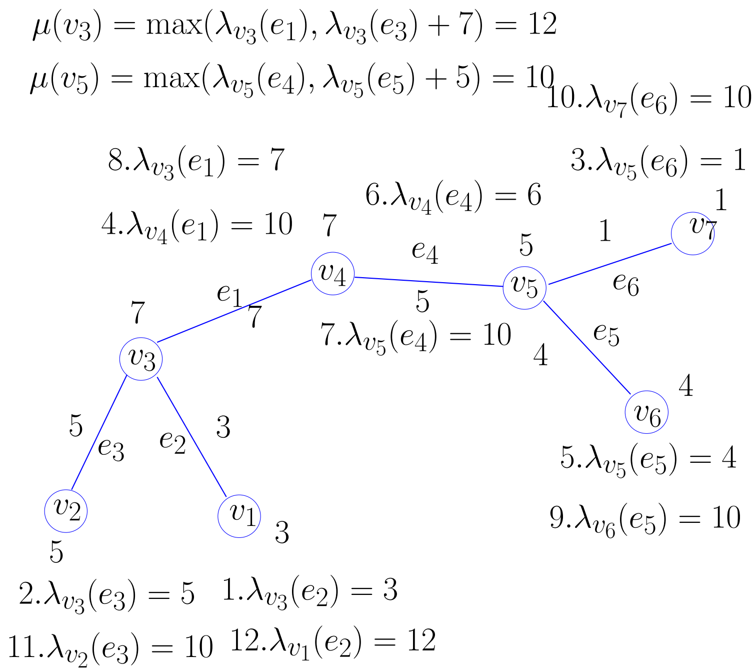


Figure 2.14: Computing $\mu(v) = \max\{\lambda_v((v, x_1)), \lambda_v((v, x_2)) + w(v)\}$ and the minimal $\min_{v \in V} \mu(v) = \mathrm{cs}(T)$ gives an optimal strategy at least for unit weighted trees.

Altogether, we have $\mu(v) = \mathrm{cs}(T_v)$ and $\min_{v \in V} \mu(v) = \mathrm{cs}(T)$. For the movements of the agents we choose the vertex v that attains a minimum $\mu(v)$ and apply a strategy as induced by the values λ_y. We traverse T_v in increasing order of the values λ_y.

For example, in Figure 2.14 $\mu(v_5) = 10$ gives the minimal number of agents required and we start with 10 agents in v_4 w.r.t. decreasing numbers λ_{v_5}. Thus, first 1 agenst move along e_6 and back to v_5, then 4 agents move along e_5 and back to v_5. After that 10 agents move along e_4 and so on.

Theorem 27 *On optimal contiguous strategy for a unit weighted tree $T = (V, E)$ can be computed in $O(n)$ time and space.*

Proof. The number of message required is given by the above considerations. For calculating the messages (and also the values $\mu(x)$) afterwards, we only have to register the greatest three entries $\lambda_v(e)$ for any v. This can be done successively. For any new message we can adjust the greatest three entries in constant time. □

2.2.7 Lower and upper bound for the contiguous search

For a given tree, T_n with n nodes we are asking for the $\max_n \mathrm{cs}(T_n) =: \mathrm{cs}(n)$. For convenience we consider the unit weighted case, where all weights are equal to 1. We will prove the following Theorem.

Theorem 28 *For unit weights and for any number of vertices n, we have $\lfloor \log_2 n \rfloor - 1 \leq cs(n) \leq \lfloor \log_2 n \rfloor$.*

We prove each inequality of the Theorem separately by the following lemmata:

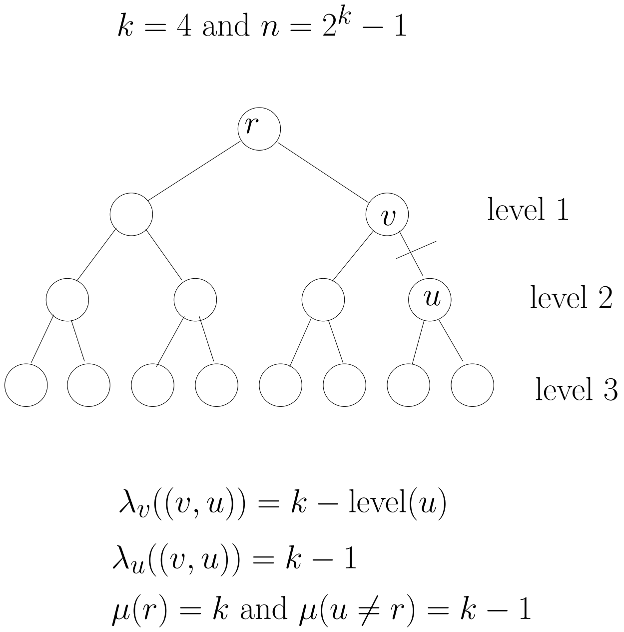


Figure 2.15: For $k = 4$ and $n = 2^k - 1$ and T_n as the full binary tree, we conclude $\text{cs}(T_n) = k - 1$ which gives the bound.

Lemma 29 *For every $n \geq 1$ we find trees T_n with $cs(T_n) \geq \lfloor \log_2(\frac{2}{3}(n+1)) \rfloor \geq \lfloor \log_2 n \rfloor - 1$.*

Proof. We consider a rooted tree T with root r and for any vertex u let the *level of* u denote the distance from r to u. If n equals $2^k - 1$ we choose a complete binary tree and show that $\text{cs}(T_n) = k - 1 = \log_2(n+1) - 1 \geq \log_2 \lfloor (\frac{2}{3}(n+1)) \rfloor$ agents are required by considering the values $\lambda_v(e)$. See also Figue 2.15.

- We have $\lambda_v((v,u)) = k - i$ and $\lambda_u((v,u) = k - 1$, for any vertex u of level $i > 0$ and its parent node v w.r.t. r. This can be easily seen by induction. The last value stem from the fact that we have to clean a complete tree with $2^{k-1} - 1$ vertices by starting from the root node.
- We have $\mu(u) = k - 1$ for any $u \neq r$ and $\mu(r) = k$, which gives the bound.

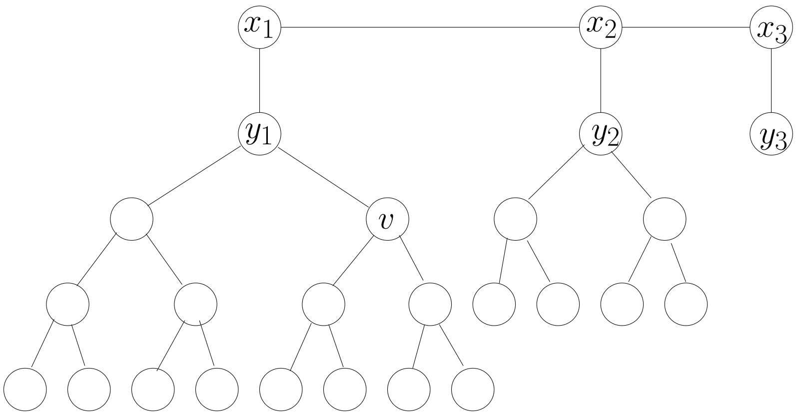
Now, for $n \neq 2^k - 1$ consider the binary representation $n = \sum_{i=1}^{r} 2^{\alpha_i}$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_r$. For example consider $n = 11010$ in binary representation with $\alpha_1 = 4$,$\alpha_2 = 3$, $\alpha_3 = 2$. We build a chain with vertices $x_1, x_2, \ldots, x_r$ and for any x_i we build an edge to a complete binary tree T_{α_i} of size $2^{\alpha_i} - 1$ as depicted in Figue 2.16.

This means that we have n vertices in total. We conclude that α_1 agents are required. This holds if we start somewhere outside T_{α_1} because we visit the root of T_{α_1} at some point. If we start inside T_{α_1} (for example in a leaf) we require $\alpha_1 - 1$ agents for T_{α_1} at most but at the root node y_i of T_{α_1} we can assume that we have to place an additional agent that blocks the recontamination from x_1.

For this we assume that we require at least $\alpha_1 - 1 = \lambda_{y_1}((y_1, x_1))$. In our example this is the case because cleaning T_{α_2} from the root requires $\alpha_2 = \alpha_1 - 1$ agent. (If this is not the case $\alpha_1 - 1$ agents are enough in total, but also n is small enough in this case so that we can also conclude $\alpha_1 - 1 \geq \lfloor \log_2(\frac{2}{3}(n+1)) \rfloor$ which is an Exercise for the cases $\alpha_2 \leq \alpha_1 - 2$).

Altogether in the above case, we have $2^{\alpha_1} - 1 < n < 2^{\alpha_1+1} - 1$ and require $\text{cs}(T_n) = \alpha_1 \geq \log_2(n+1) - 1 \geq \log_2 \lfloor (\frac{2}{3}(n+1)) \rfloor$ agents in total which gives the conclusion. $\square$

$$n = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 11010$$



$$\lambda_{y_1}((v, y_1)) = \alpha_1 - 1$$
$$\lambda_{y_1}((x_1, y_1)) = \alpha_2 + 1 = \alpha_1$$

Figure 2.16: A tree T_n with $n = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 26$ vertices, requires α_1 agents.

Exercise 14 *Discuss the remaining case in the above proof. That is $\alpha_2 < \alpha_1 + 1$. Consider $\alpha_2 = \alpha_1 + 2$ and $\alpha_2 < \alpha_1 + 2$ separately.*

On the other hand we show that $\lfloor \log_2 n \rfloor$ agents are always sufficient.

Lemma 30 *For every $n \geq 1$ and unit weights, $\lfloor \log_2 n \rfloor$ agents are sufficient for a contiguous search strategy.*

Proof. We consider a tree T_r with n vertices and $\mu(r) = \text{cs}(T)$. Now we simplify this so that it becomes a complete binary tree T'_r w.r.t. r with $\text{cs}(T_r) = \text{cs}(T'_r)$ by the following rules, which will be applied until none of them is applicable any more. The children/parent relation in the tree is considered w.r.t. r.

1. For a node x and its $d > 2$ children $x_1, x_2, \ldots, x_d$ ordered by $\text{cs}(T_r(x_i)) \geq \text{cs}(T_r(x_{i+1}))$ remove all $T_r(x_i)$ for $i > 2$.

2. For a node x with two children x_1 and x_2 and $\text{cs}(T_r(x_1)) > \text{cs}(T_r(x_2))$, remove $T_r(x_2)$.

3. For a node $x \neq r$ with only one child x_1, remove x and connect x_1 to the parent of x.

4. If there are more than two vertices left, and r has only one child x_1, remove x_1 and connect the children of x_1 to r.

First, the number of agents required for T'_r and T_r are the same, because the computation of $\mu(r)$ in T_r makes use of eaxctly the same values. Note that the weights of the vertices are restricted to one, therefore rule 2. is also correct by $\text{cs}(T_r(x_1)) \geq \text{cs}(T_r(x_2)) + 1$. Cancelling a vertex with one child has no influence.

Second, we show that T'_r is a complete binary tree rooted in r. The first rule and the second rule returns a tree that has internal nodes with at most 2 children. Rule three deletes internal nodes with one child except for the root. Rule 4 make the root have 2 or 0 children.

Thus, we have a binary tree whose internal nodes have degree excactly 2. Finally, we show that the tree is complete. Let x be a node such that the subtree T'_x at x is not complete and there is no other subtree in T'_x with this property. This means that the children x_1 and x_2 of x in T'_r define complete subtree T'_{x_1} and T'_{x_2} of different size. Thus, rule 2 can be applied which gives a contradiction. □