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Theoretical Aspects of Intruder Search

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The manuscript will be successively extended during the lecture in the Wintersemester. Hints and comments for improvements can be given to Elmar Langetepe by E-Mail `elmar.langetepe@informatik.uni-bonn.de`. Thanks in advance!

5.6.4 Recursions

Our next task is to make the integrals in 5.24 and 5.25 disappear by iterated substitution, and then, to substitute variable l with concrete values l_1 (resp. l_2), based on Lemma 64.

Let us define, for positive real numbers $x_0, \ldots, x_n$, the auxiliary expression

$$I_n(x_n) \;:=\; \int_0^{x_n} \frac{1}{F_0(x_{n-1})} \int_0^{x_{n-1}} \frac{1}{F_0(x_{n-2})} \cdots \int_0^{x_1} \frac{1}{F_0(x_0)}\, \mathrm{d}x_0 \ldots \mathrm{d}x_{n-1}.$$

By induction on n one quickly shows

$$I_n(x_n) \;=\; \frac{1}{n!}\,\frac{1}{\cos^n \alpha}\,\big(\ln(\frac{A+\cos\alpha x_n}{A})\big)^n$$

since $F_0(x) = A + \cos\alpha\, x$; compare the discussion after formula 5.18. Now we iteratively substitute G_j in formula 5.24 and obtain

$$G_{j+1}(l) \;=\; \frac{\phi_0(l_2)}{F_0(0)} \sum_{\nu=0}^{j} (-1)^\nu\, I_\nu(l)\, \chi_{j-\nu}(l_2) \;+\; (-1)^{j+1} I_{j+1}(l). \tag{5.26}$$

By definition of l_1, according to formula 5.15, we have $\ln(\frac{A+\cos(\alpha)l_1}{A}) = 2\pi \cot\alpha$, so that setting $l = l_1$ in formula 5.26 leads to

$$G_j(l_1) \;=\; \frac{\phi_0(l_2)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!}\,\big(\frac{2\pi}{\sin\alpha}\big)^\nu\, \chi_{j-1-\nu}(l_2) \tag{5.27}$$

where, for convenience, $\chi_{-1}(l_2) := \frac{F_0(0)}{\phi_0(l_2)}$. We observe that this formula is also true for $j = 0$. Multiplying both sides by $F_0(l_1)$, and re-substituting 5.23, results in

$$F_j(l_1) \;=\; \frac{F_0(l_1)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!}\,\big(\frac{2\pi}{\sin\alpha}\big)^\nu\, \phi_{j-1-\nu}(l_2) \tag{5.28}$$

where $\phi_{-1}(l_2) := F_0(0)$.

In a similar way we solve the recursion in 5.25, using

$$\int_0^{x_n} \frac{1}{\phi_0(x_{n-1})} \int_0^{x_{n-1}} \cdots \int_0^{x_1} \frac{1}{\phi_0(x_0)}\, \mathrm{d}x_0 \ldots \mathrm{d}x_{n-1} \;=\; \frac{1}{n!}\,\frac{1}{\cos^n \alpha}\,\big(\ln(\frac{x_n}{l_1})\big)^n$$

and $\ln(\frac{l_2}{l_1}) = \alpha \cot\alpha$, according to formulae 5.15 and 5.16. One obtains, after substituting $l = l_2$,

$$\phi_j(l_2) \;=\; \frac{\phi_0(l_2)}{\phi_0(l_1)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!}\,\big(\frac{\alpha}{\sin\alpha}\big)^\nu\, \hat{F}_{j-\nu}(l_1) \tag{5.29}$$

where $\hat{F}_0(l_1) := \phi_0(l_1)$ and $\hat{F}_{i+1}(l_1) := F_{i+1}(l_1)$.

5.7 Generating functions

In order to solve the cross-wise recursions 5.28 and 5.29 for the numbers $F_j(l_1)$ we are interested in, we define the generating functions

$$F(Z) := \sum_{j=0}^{\infty} F_j\, Z^j \quad \text{and} \quad \phi(Z) := \sum_{j=0}^{\infty} \phi_j\, Z^j$$

where $F_j := F_j(l_1)$ and $\phi_j := \phi_j(l_2)$, for short. From 5.28 we obtain

$$F(Z) \;=\; \frac{F_0}{F_0(0)}\, e^{-\frac{2\pi}{\sin\alpha}Z}\,\big(Z\,\phi(Z) + F_0(0)\big), \tag{5.30}$$

and from 5.29,

$$\phi(Z) \;=\; \frac{\phi_0}{\phi_0(l_1)}\, e^{-\frac{\alpha}{\sin\alpha}Z}\,\big(Z\,F(Z) - F_0 + \phi_0(l_1)\big). \tag{5.31}$$

Both equalities can be verified by plugging in expansions of the exponential functions, using $e^W = \sum_{j=0}^{\infty} \frac{W^j}{j!}$, computing the products, and comparing coefficients. Now we substitute 5.31 into 5.30, solve for $F(Z)$, divide both sides by F_0 and expand by $e^{\frac{2\pi+\alpha}{\sin\alpha}}$ to obtain the surprisingly simple formula

$$\frac{F(Z)}{F_0} \;=\; \frac{e^{vZ} \;-\; r\,Z}{e^{wZ} \;-\; s\,Z}, \tag{5.32}$$

where v, r, w, s are the following functions of α:

$$\begin{aligned} v \;&=\; \frac{\alpha}{\sin\alpha} \quad \text{and} \quad r \;=\; e^{\alpha\cot\alpha} \\ w \;&=\; \frac{2\pi+\alpha}{\sin\alpha} \quad \text{and} \quad s \;=\; e^{(2\pi+\alpha)\cot\alpha}. \end{aligned} \tag{5.33}$$

We recall that α denotes the angle of the fighter's velocity vector, given by $\alpha = \cos^{-1}(1/v)$.

It is possible to expand the inverse of the denominator in 5.32 into a power series. This leads to interesting expressions for the F_j; but how to derive their signs seems not obvious.

5.8 Singularities

Both, numerator and denominator of function $F(Z)$, are analytic on the complex plane. Thus, singularities of F can arise only from zeroes of the denominator,

$$e^{wZ} - sZ. \tag{5.34}$$

This equation has received some attention in the area of delay differential equations. We need a simple fact that has been published in [**?**].

Lemma 65 *For $s < ew$, equation 5.34 has an infinite, discrete set of conjugate complex zeroes none of which are real.*

As the fighter' speed v increases from 1 to infinity, angle α of her velocity vector increases from 0 to $\pi/2$. This causes the ratio s/w in 5.34 to decrease from infinity to zero. Exactly at $v_c = 2.6144\ldots$ does the equality $s/w = e$ hold. Hence, for $v > v_c$ we have $s < ew$, so that the denominator of $F(Z)$ has infinitely many strictly complex zeroes. The root classification given in [**?**] shows that only few of them can cancel out with the numerator of 5.32. A complete proof of this fact can also be found in [**?**], Lemma 10-13.

We obtain the following from the above.

Lemma 66 *If $v > v_c$ then function $F(Z)$ has an infinite, discrete set of complex poles none of which are real.*

We are going to apply the following result from complex function theory; see, for example, [**?**] p. 240.

Theorem 67 (Pringsheim) *Let* $H(Z) = \sum_{n=0}^{\infty} a_n Z^n$ *be a power series with finite radius of convergence,* R*. If* $H(Z)$ *has only non-negative coefficients* a_n*, then point* $Z = R$ *is a singularity of* $H(z)$*.*

Now we are ready to prove Theorem 59.

Proof.[Proof of Theorem 59] Suppose that the fighter's speed v is larger than $v_c \approx 2.6144$. By Lemma 66, $F(Z)$ does have a discrete set of poles, and therefore, a finite radius of convergence, R. If all coefficients F_j of $F(Z)$ were positive, R would be a singularity of $F(Z)$, by Pringsheim's theorem; but we know from Lemma 66 that there are no real singularities. Thus, there must be coefficients $F_j \leq 0$, and we conclude from Lemma 64 that the fighter succeeds in containing the fire. □