Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Graphs and Trees

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Organisation

- Lecture: Tuesday 16:15 to 17:45
- Exercise groups: Starting 28/29th
  Wednesday: 14:15-15:45
  Thursday: 10:15-11:45
- Sign in
- Manuscipt on the webpage
- Slides on the webpage
- Exercises
- Today: Introduction

Repetition: Main problems and intention

- Evader/Intruder versus Searcher/Guard
- Escaping/Intruding versus Catching/Avoidance
- Game, Competition
- Different Scenarios: Environment, Facilities, Goal, Model
- Discrete, Continuous, Geometry, Combinatorics
- Interpretation: Possible Position of the Intruder, Decontamination, Firefighting

Repetition: Theoretical Aspects

- Algorithmic track
- Computational complexity
- Correctness or Failure
- Efficiency
- Optimality
- Prerequisites: Algorithms, Datastructure, Analysis, Complexity, Computability
- Models, Methods, Proof Techniques, Tools

Repetition: Different Examples

1. Optimal-Closing-Sequence against Intruder
   - Saving maximum area by closing doors in polygon
   - NP-hard, Reduction of Subset-Sum
2. Catching-an-Evader
   - Grid-Graph game, Evader moves, k stationary Guards
   - Correctness $k = 1$ impossible, $k = 2$, optimal solution by ILP
3. Enclosing-a-Fire
   - Expanding circle in the plane, Build a barrier with speed $v > 1$
   - Barrier Curves: Circle arround origin of fire, $v \geq 2\pi$ tight
4. Discrete Fire-Figthing-Curve
   - Grid-Graph, Fire spreads after n steps, barrier cell after b steps
   - Conjecture, $b < \frac{n-1}{2}$ tight bound! Simulation!

- Graph $G = (V, E)$, degree d, root r, p firefigther per step
- Different models: Intruder-Search/Firefigthing
- Complexity
- Optimal Algorithms
- Approximation

Graph of degree 3

- NP-complete
- Simple Algorithm for root vertex of degree 2
- Defending a vertex
- $\texttt{dist}(u, r)$ length of a shortest path from r to u
- V_1 vertices of degree 1, V_2 vertices of degree 2
- V_c vertices of degree 3 that belong to a cycle
- Example

Path Strategy

Lemma 6: Vertex $u \in V_1 \cup V_2$ can be enclosed in time $\mathtt{dist}(u, r) + 1$ and only $\mathtt{dist}(u, r) + 1$ vertices are on fire. Vertex $u \in V_c$ can be enclosed in time $\mathtt{dist}(u, r) + C(u) - 1$ and only $\mathtt{dist}(u, r) + C(u) - 1$ vertices are on fire.

Proof! Constructive!

Graph of degree 3, root vertex degree 2

Optimal Strategy:

$$f(u) := \begin{cases} \texttt{dist}(u,r)+1 & : \quad \text{if } u \in V_1 \cup V_2 \\ \texttt{dist}(u,r)+C(u)-1 & : \quad \text{if } u \in V_c \setminus V_2 \\ \infty & : \quad \text{otherwise} \end{cases}$$

Find a vertex u with $f(u) = \min_{x \in V} f(x)$. Enclose this vertex by the path strategy.

Graph of degree 3, root vertex degree 2

Structural Property

Lemma 7: For a setting $(G, r, 1)$ where G has maximal degree 3 and root r has degree ≤ 2 there is always an optimal protection strategy that protects the neighbor of a contaminated vertex in each time step.

Proof!

- Trivial for degree r is 1
- Degree of r is 2, mimimal counterexample
- I. one of the neighbors of r first, II. not a neighbor of r

Graph of degree 3, root vertex degree 2

Optimal Strategy:

$$f(u) := \begin{cases} \texttt{dist}(u,r)+1 & : \quad \text{if } u \in V_1 \cup V_2 \\ \texttt{dist}(u,r)+C(u)-1 & : \quad \text{if } u \in V_c \setminus V_2 \\ \infty & : \quad \text{otherwise} \end{cases}$$

Find a vertex u with $f(u) = \min_{x \in V} f(x)$. Enclose this vertex by the path strategy.

Theorem 8: For a problem instance $(G, r, 1)$ of a graph G of maximum degree 3 and a root vertex of degree 2 the above strategy is optimal.

Proof!

Theorem 8: For a problem instance $(G, r, 1)$ of a graph G of maximum degree 3 and a root vertex of degree 2 the above strategy is optimal.

Proof:

- Last burning vertex u
- $u \in V_1, V_2$, by construction
- $u \in V_c$, neighbors n_1, n_2, n_3, and n_1 on fire
- Also n_2 on fire: u, n_1 and n_2 on a cycle, contradiction!
- n_2 and n_3 are protected.
- Another neigbor of n_2 or n_3 is on fire, say p of n_2
- Otherwise: protect u one step earlier
- u, n_2, p build the cycle

Graph of degree 3, root vertex degree 2

Theorem 9: For a problem instance $(G, r, 1)$ of a graph $G = (V, E)$ of maximum degree 3 and a root vertex of degree 2 the decision problem can be solved in polynomial time and the maximum number of vertices that can be saved is $|V| - \min_{x \in V} f(x)$.

Proof: Compute the values in polynomial time!

NP-Competeness for general graphs

Theorem 10: The firefighter decision problem for graphs is NP-hard.

Proof:

- k-Clique reduction to the decision problem
- Construct Graph $G' = (V', E')$ from Graph $G = (V, E)$
- Vertex s_v for every $v \in V$, Vertex s_e for every $e \in E$
- Edges (s_v, s_e) and (s_u, s_e) for every edge $e = (u, v)$
- Root vertex r and $k - 1$ colums of k vertices $v_{i,j}$
- Connect the layer from left to right and to all s_v
- Example Blackboard!

NP-Competeness for general graphs

Theorem 10: The firefighter decision problem for graphs is NP-hard.

Proof:

1. k-Clique exists
   - Save k vertices of the k Clique
   - Saves $k' = k + \binom{k}{2} + 1$ vertices
2. Saving $k' = k + \binom{k}{2} + 1$ vertices?
   - Before step k: Saving s_e or $a_{i,j}$ needless, only one
   - Saving k vertices s_v.
   - k' possible, if k-Clique exists
   - Another one in the last step

NP-Competeness for general graphs

Theorem 11: For a problem instance $(T, r, 1)$ of a rooted tree $T = (V, E)$ the greedy strategy gives a $\frac{1}{2}$ approximation for the optimal number of vertices protected. This bound is tight.

Proof:

1. Example for $\frac{k+1}{2(k-1)} \mapsto \frac{1}{2}$
2. Tightness
   - Greedy versus $\texttt{opt}$, time steps: Savings
   - $\texttt{opt}_A$ not better than greedy, $\texttt{opt}_B$ better than greedy.
   - $2S_G \geq \texttt{opt}_A + \texttt{opt}_B$
   - Greedy competes with $\texttt{opt}_A$ at the start
   - Moment where Greedy is worse that $\texttt{opt}$
   - $\texttt{opt}_B$ choose v, depth l, also greedy can choose l or greedy has chosen a predecessor of v before $\Rightarrow$ greedy saves at least the vertices of $\texttt{opt}_B$ before

Firefighter Decision Problem (Protection by k guards):
Instance: A Graph $G = (V, E)$ of degree d with root vertex r and p firefigther per step and an integer k.
Question: What is the strategy that saves a maximum number of vertices by protecting k vertices in total?

Efficient Algorithm for Trees

Dynamic Programming Approach: Place k guards! Structural Property!

Lemma 12: For any optimal strategy for an instance of the firefigther decision problems on trees (protection by k guards, saving k vertices) the vertex defended at each time is adjacent to a burning vertex. There is an integer l, so that all protected vertices have depth at most l, exactly one vertex p_i at each depth is protected and all ancestors of p_i are burning.

- Time step t, place guard with non-burning neighbor
- Placement closer to the root improves strategy
- depth t at step t, inductively!

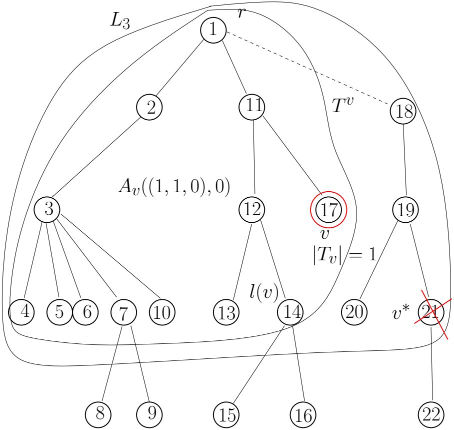
Efficient Algorithm for Trees

Dynamic Programming Approach: Place k guards!

- Lemma 12: Guards in depth $1, 2, \ldots, k$
- L_k, vertices of T with depth $\leq k$
- Order for the processing: Subproblems!
- Preorder of the graph! *to the left*, *rightmost*
- $l(v)$, vertex to the left of v
- T_v subtree at v
- T^v tree with vertices from L_k to the left of v, including v
- Recursion more general: Vector $X \in \{1, 0\}^k$
- $X(j) = 1$ place guard in step j, $X(j) = 0$ n guard in step j
- $A_v(X)$: Optimal strategy for X in T^v, based on $T^{l(v)}$
- Recursion!

Efficient Algorithm for Trees

Problem! No two guards on a single path! Set guard after depth i!



$|T_v| + A_{l(v)}((1,0,0),1)$ or $A_{l(v)}((1,1,0),0)$

$$A_v(X, i) :=$$

Optimal protection number in T^v for strategy that sets the guards w.r.t. entries of X and no guard is set on the path from r to v at depth $\leq i$

Efficient Algorithm for Trees

Theorem 13: Computing the optimal protection strategy for k guards on a tree T of size n can be done in $O(n2^k k)$ time.

$$A_v(X,i) :=$$
$$\max \left\{ \begin{array}{l} A_{l(v)}(X, \min(\mathtt{d}(v)-1, i)) \\ \\ [X(\mathtt{d}(v)) = 1 \ \& \ \mathtt{d}(v) > i] \cdot \big(|T_v| + A_{l(v)}(X^v, \mathtt{d}(v)-1)\big) \end{array} \right\}$$

- Compute L_k, $l(v)$, $|T^v|$ in linear time!
- Traverse the vertices of L_k from left to right
- At most $n \times 2^k \times k$ entries $A_v(X,i)$
- n stands for v, 2^k stands for X, k stands for i.

Efficient Algorithm for Trees

Corollary 14: Computing a strategy for a tree T of size n that saves at least k vertices can be done in $O(n2^k k)$ time.

- Run above algorithm for $i = 1, \ldots, k$
- Sufficient!
- $\sum_{i=1}^{k} i2^i n \leq kn \sum_{i=1}^{k} 2^i = (2^{k+1} - 2)kn$