Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Geometric Firefighting

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Connnected Search vs. non-connected search

- Non-connected, other rules!
- Differ in a factor of 2

1. Place a team of p guards on a vertex.
2. Move a team of m guards along an edge.
3. Remove a team of r guards from a vertex.

Connnected Search vs. non-connected search

D_k denote a tree with root r of degree three and three full binary trees, B_{k-1}, of depth $k-1$ connected to the r.

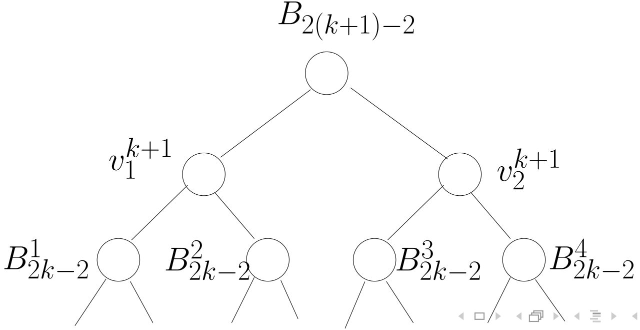
Lemma 31: For the graph D_k, we conclude $\mathtt{cs}(D_k) = k+1$.

- Consider T_1, T_2 and T_3 at r!
- At most $k+1$
- At least $k+1$

D_k denote a tree with root r of degree three and three full binary trees, B_{k-1}, of depth $k-1$ connected to the r.

Lemma 32: For D_{2k-1} we conclude $s(D_{2k-1}) \leq k+1$.

- $k = 1$ is trivial. So assume $k > 1$
- Place one agent at the root r and successively clean the copies of B_{2k-2} by k agents
- This is shown by induction!



Connnected Search vs. non-connected search

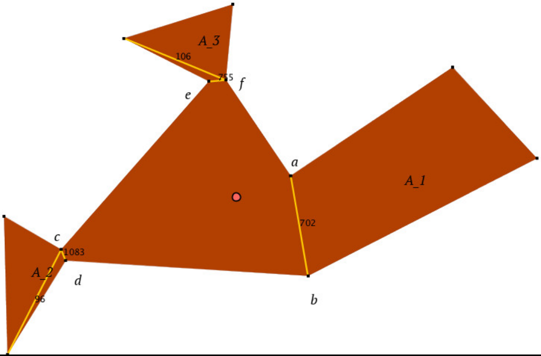
Corollary 33: There exists a tree T so that $cs(T) \leq 2s(T) - 2$ holds.

$T = D_{2k-1}$, $s(D_{2k-1}) \leq k + 1$, $\mathrm{cs}(D_{2k-1}) = 2k$

$$\frac{cs(T)}{s(T)} < 2 \text{ for all trees } T\,.$$

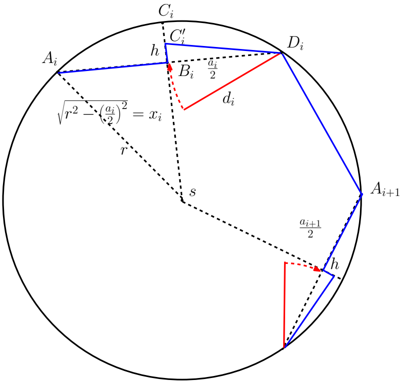
- Intruder/Contam. constant speed, exclude fire, fences
- First, inside a polygon, single fire source,
- Build linear barriers with speed B, build barriers successively

Instance: Simple polygon, fire spreads from $s \in P$ with speed 1, m line segment *barriers*, b_i successively constructed with speed B.
Output: Valid sequence of barriers constructed successively, area blocked from the fire is maximized.



Geometric firefigthing, simple polygon

- **Theorem 1:** Computing an optimal-enclosement-sequence is NP-hard.
- Approximation hard!
- Our goal: Polynomial time constant approximation!



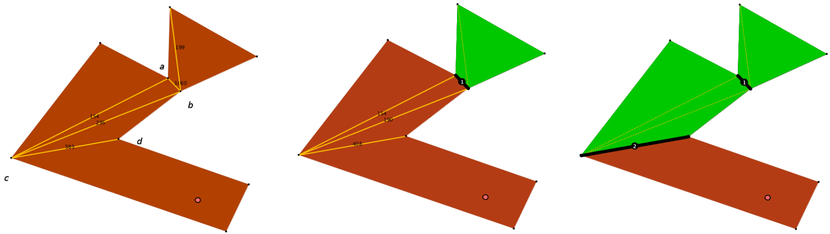
Geometric firefigthing, simple polygon, approximation

- General scheduling algorithm, working with profits
- 0.086-approximation of optimal profit (area).
- Non-intersecting barriers, is an application!
- Intersection is more difficult!
- Framework: Set of jobs $b_1, b_2, \ldots, b_m$
- Duration d_i, starting time s_i (start before s_i!)
- Algorithm: n steps schedule $J_n = (b_{n_1}, b_{n_2}, \ldots, b_{n_{l_n}})$
- Size l_n, n jobs considered, s'_{n_k} precise starting time
- Valid: $\sum_{k=1}^{j} s'_{n_k} + d_{n_k} \leq s_{n_{j+1}}$ for $j = 1$ to $l_n - 1$
- Job b_i contribute with a profit A_i to overall profit A

Geometric firefigthing, simple polygon, approximation

- Profits might overlap! $A_i \cup A_j \neq \emptyset$
- Schedule: $J_n = (b_{n_1}, b_{n_2}, \ldots, b_{n_{l_n}})$
- $b_j \notin J_n$, current profit! Can decrease!

$$A_j(J_n) := A_j \setminus \left(\bigcup_{b_{n_k} \in J_n} A_{n_k} \right)$$



Approximation scheme: GlobalGreedy

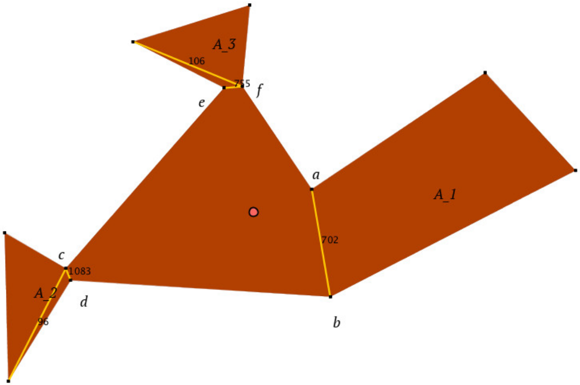
- Empty schedule J_0, constant $\mu < 1$
- Sort remaining jobs b_j by $\frac{A_j(J_n)}{d_j}$, process largest!

1. b_j can be scheduled somewhere in J_n. Insert b_j: J_{n+1}
2. b_j cannot be processed, overlaps with jobs in J_n.
   Find sequence in J_n that overlaps:
   1. Profits of these jobs smaller than μ times $A_j(J_n)$.
   2. b_j can be scheduled after deletion of the jobs.

   Then build J_{n+1} with b_j.
   Deleted jobs will never be processed again.
3. No such sequence exists in J_n. Reject b_j!

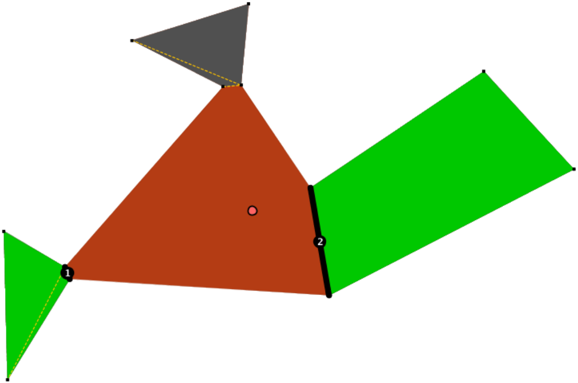
Color scheme: Green profit/jobs (inserted), grey profit/jobs (deleted afterwards)
All profits (universe) red in the beginning!

Approximation scheme: GlobalGreedy Example



- $b_1 = (a, b)$, $b_2 = (c, d)$, $b_3 = (e, f)$, $|b_1| = 3$, $|b_2| = 0.3$, $|b_3| = 0.5$, speed 2, $\mu = 0.2$
- $A_1 = 1053$, $A_2 = 162.45$, $A_3 = 188.75$, $d_P(s, a) = 1.8$
- $p_1 = 702 = \frac{2A_1}{3}$, $p_2 = \frac{2A_2}{0.3} = 1083$, $p_3 = \frac{2A_3}{0.5} = 755$

Approximation scheme: GlobalGreedy Example



- $p_1 = 702 = \frac{2A_1}{3}$, $p_2 = \frac{2A_2}{0.3} = 1083$, $p_3 = \frac{2A_3}{0.5} = 755$
- $J_2 = (b_2, b_3)$, $(0.3 + 0.5 + 3)/2 = 1.9 > d_P(s, a)$
- $\mu \cdot A_1 > A_3$, $(0.3 + d(a, b))/2 < d(s, a)$, $J_3 = (b_2, b_1)$.

- $J_n(grey)$ and $J_n(green)$ colored green/grey during the construction of J_n.
- J_n': All jobs that where inserted, green/grey

Lemma 52: $J_m(grey) \leq \frac{\mu}{1-\mu} J_m(green)$.

- By induction on the jobs processed during GlobalGreedy
- Base: Holds for J_0
- Assume that the lemma holds after n steps for J_n. Consider step $n+1$.

Lemma 52: $J_m(grey) \leq \frac{\mu}{1-\mu} J_m(green)$.

- Inductive step: $n \to n+1$, consider b_j with $A_j(J_n)$
- No job deleted (Rules 1.,3.): Only green can increases!

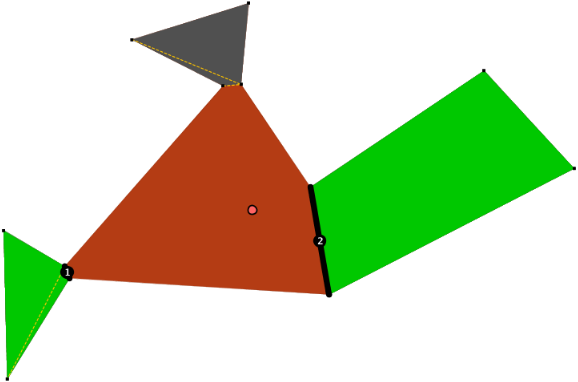
$$J_n(grey) = J_{n+1}(grey) \leq \frac{\mu}{1-\mu} J_n(green) \leq \frac{\mu}{1-\mu} J_{n+1}(grey).$$

- Rule 2., some jobs deleted: smaller μ times $A_j(J_n)$

$$\begin{aligned}
\frac{\mu}{1-\mu} J_{n+1}(green) &\geq \frac{\mu}{1-\mu}(J_n(green) + (1-\mu)A_j(J_n)) \\
&\geq \frac{\mu}{1-\mu} J_n(green) + \mu A_j(J_n) \\
&\geq J_n(grey) + \mu A_j(J_n) \geq J_{n+1}(grey),
\end{aligned}$$

Relationsship to optimal sequence, $J_{\mathtt{opt}}$

- Green and grey profits/jobs
- $J_{\mathtt{opt}}$: Red profits finally not colored green or grey, colored blue!
- Example: Job b_3 will be scheduled, no blue color!
- Assign, blue profit to the first job in $J_{\mathtt{opt}}$, that covers profit!
- $|J_{\mathtt{opt}}| \leq J_m(blue) + J_m(green) + J_m(grey)$.



Relationsship to optimal sequence, J_{opt}

- Green, grey, blue profits/jobs are disjoint!
- Expresse blue profit in terms of grey and green profit
- Payment scheme! Green/grey (J'_m) pay to blue jobs!
- $b_i \in J'_m$ gets unique execution time! Pays to some $b_j \in J_{\mathrm{opt}}$!

1. If the execution interval of $b_j \in J_{\mathrm{opt}}$ is fully included in the execution interval of $b_i \in J'_m$, the job b_i pays its green or grey profit times $\frac{d_j}{d_i} < 1$ to b_j.
2. If the execution interval of $b_j \in J_{\mathrm{opt}}$ overlaps with the execution interval of $b_i \in J'_m$, the job b_i pays its green or grey profit times $\frac{1}{\mu}$ to b_j.

Relationsship to optimal sequence, $J_{\mathtt{opt}}$

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Lemma 53: Any single green or grey job from J'_m pays in total at most $1 + \frac{2}{\mu}$ times its profit to the blue jobs.

Lemma 54: Any single blue job from $J_{\mathtt{opt}}$ achieves at least a payment in the size of its blue profit from the green and grey jobs.

$$J_m(blue) \leq \left(1 + \frac{2}{\mu}\right)\left(J_m(green) + J_m(grey)\right).$$

Relationsship to optimal sequence, $J_{\texttt{opt}}$

Lemmate 53/54:

$$J_m(blue) \leq \left(1 + \frac{2}{\mu}\right)\left(J_m(green) + J_m(grey)\right).$$

Lemma 52:

$$J_m(grey) \leq \frac{\mu}{1-\mu} J_m(green).$$

$$\begin{aligned}
|J_{\texttt{opt}}| &\leq J_m(blue) + J_m(green) + J_m(grey) && (1)\\
&\leq \left(2 + \frac{2}{\mu}\right)\left(J_m(green) + J_m(grey)\right) && (2)\\
&\leq \frac{2(\mu+1)}{\mu}\left(J_m(green) + \frac{\mu}{1-\mu} J_m(green)\right) && (3)\\
&\leq \frac{2(\mu+1)}{\mu}\frac{1}{1-\mu} J_m(green) && (4)\\
&\leq 2\frac{\mu+1}{\mu(1-\mu)} J_m(green) \leq 2\frac{\mu+1}{\mu(1-\mu)} |J_m|. && (5)
\end{aligned}$$

Relationsship to optimal sequence, $J_{\texttt{opt}}$

$$|J_{\texttt{opt}}| \;\leq\; 2\frac{\mu+1}{\mu(1-\mu)}|J_m|\,.$$

Minimize: $f(\mu) := 2\frac{\mu+1}{\mu(1-\mu)}$
By $\mu = \sqrt{2}-1$ this gives $f(\mu) = 6+4\sqrt{2} \approx 11.657$

Theorem 55: For the geometric firefighter problem inside a simple polygon with non-intersecting barriers there is an approximation algorithms that saves at least $\frac{1}{6+4\sqrt{2}} = \frac{3}{2} - \sqrt{2} \approx 0.086$ times the area of the optimal barrier solution.

- Applicable to the barrier construction problem!
- Intersections, dependencies between barriers!

Relationsship to optimal sequence, J_{opt}

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Lemma 54: Any single blue job from J_{opt} achieves at least a payment in the size of its blue profit from the green and grey jobs.

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Lemma 53: Any single green or grey job from J'_m pays in total at most $1 + \frac{2}{\mu}$ times its profit to the blue jobs.

- $b_i \in J'_m$ has fixed execution interval I_i with start- and endtime
- Interval of $b_j \in J_{\texttt{opt}}$ fully inside I_i: $\frac{d_j}{d_i}$, sums up to at most 1 for all $b_j \in J_{\texttt{opt}}$
- Two intervals $b_j \in J_{\texttt{opt}}$ can overlap I_i: 2 times $\frac{1}{\mu}$ the profit of b_i.

Relationsship to optimal sequence, $J_{\texttt{opt}}$

1. If the execution interval of $b_j \in J_{\texttt{opt}}$ is fully included in the execution interval of $b_i \in J'_m$, the job b_i pays its green or grey profit times $\frac{d_j}{d_i} < 1$ to b_j.
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Lemma 54: Any single blue job from $J_{\texttt{opt}}$ achieves at least a payment in the size of its blue profit from the green and grey jobs.

- Blue job $b_j \in J_{\texttt{opt}}$, job has to be rejected in step $k+1$, Consider execution time interval of $b_j \in J_{\texttt{opt}}$
- Subset $\overline{J_k}$ of $J_k = (b_{k_1}, b_{k_2}, \ldots, b_{k_{l_k}})$ that *minimally* overlaps with execution interval for b_j
- Total profit $\overline{J_k}$ larger than μ times curr. red profit $A_j(J_k)$ of b_j
- Larger μ times the final blue part of b_j
- b_j less priority: $\frac{A_i(J_k)}{d_i} \geq \frac{A_j(J_k)}{d_j}$

Relationsship to optimal sequence, J_{opt}

Lemma 54: Any single blue job from J_{opt} achieves at least a payment in the size of its blue profit from the green and grey jobs.

1. Total profit $\overline{J_k}$ larger than μ times final blue profit of b_j ($\leq A_j(J_k)$)
2. b_j less priority: $\frac{A_i(J_k)}{d_i} \geq \frac{A_j(J_k)}{d_j}$

- $|\overline{J_k}| = 1$ for single job, say b_i
- $b_j \in J_{\mathrm{opt}}$ might be fully inside the execution time of b_i:

$$\text{Pay: } A_i(J_k)\frac{d_j}{d_i} \geq A_j(J_k)\frac{d_i}{d_i} = A_j(J_k)$$

- For $|\overline{J_k}| \geq 1$, execution interval of b_j overlaps with all execution intervals in $\overline{J_k}$:

$$\text{Pay: } \frac{1}{\mu} \sum_{b_i \in \overline{J_k}} A_i(J_k) \geq \frac{1}{\mu}(\mu A_j(J_k)) = A_j(J_k)$$