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Online Motion Planning

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Chapter 2

Polygonal enviroments

In this section we turn over to planar environments modelled by (a set of) simple polygons. We assume that a set of simple polygons P_i for $i = 1,\dots,k$ is given. Two polygons do neither intersect nor touch each other. The number of polygons is finite in the sense that any circle of fixed radius contains only finitely many *obstacles P_i*.

In the following sections an agent tries to escape from a labyrinth (modelled by polygons) or tries to find a target t in a polygonal environment. We assume that the agent is point-shaped and thus consider curves in the plane. From a practical point of view one might think that a physical robot somehow *follows* the corresponding curve without actually running precisely on it. Additionally, in the followning configurations, the agent has only a limited storage.

We make use of the following conventions. If the coordinates of the target are given, the task of the agent is denoted as “Navigation”. On the other hand, if the coordinates are not known the task of the agent is “Searching”. We will consider different sensor models.

Some of the following algorithms – for example the Plegde-algorithm and some Bug-Variants – have been implemented as interactive Java-Applets, see

`http://web.informatik.uni-bonn.de/I/GeomLab/Polyrobot/`

2.1 Escape from the labyrinth

The task of the agent is to escape from a polygonal environment. In the geometric sense the agent escapes, if it finally hits a circle that contains all obstacles.

The agent is point-shaped and makes use of a touch-sensor for following the wall by Left-Hand-Rule or Right-Hand-Rule. Additionally, we allow that the agent can count its total turning angles in a single ; see Figure 2.2(i). The agent realizes when it hits the enclosing circle. In this case the agent successfully escaped from the labyrinth.

2.1.1 Pledge-Algorithm

Algorithm 2.1 Pledge-Algorithmu

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1. Choose direction $\varphi$, turn agent into direction $\varphi$.
2. Move into dircetion $\varphi$, until an obstacle is met.
3. Turn right and follow the wall by Left-Hand-Rule.
4. Follow the wall, sum up the overall turning angles, until the angular counter gets zero, in this case
   GOTO (2).
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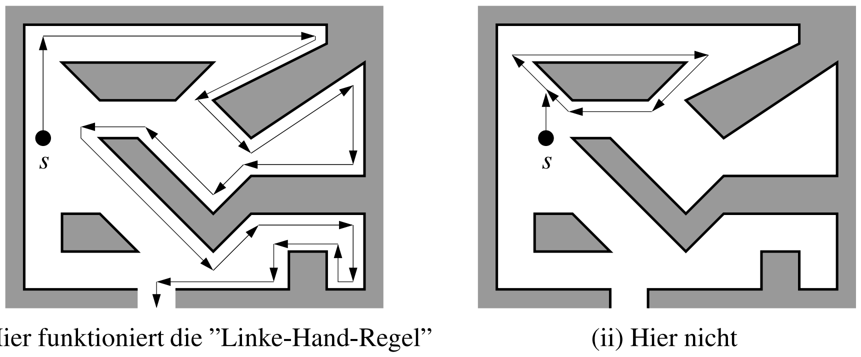


Figure 2.1: Simple strategies cannot be successful.

Note that, simple counting schemes or movements cannot succeed. For example, following the wall by Left-Hand-Rule until the agent points again into direction φ and leave the obstacle right now can result in infinite loops; see Figure 2.2(ii). Just following the boundary could let the agent stuck inside the labyrinth; see Figure 2.1.

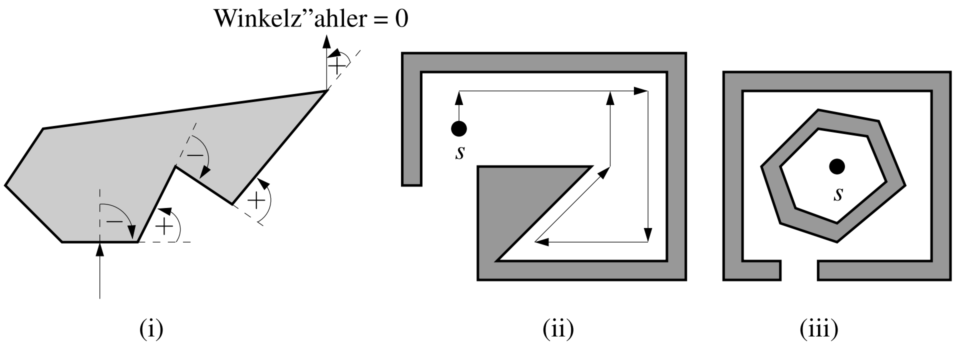


Figure 2.2: (i) Angular counter. (ii) Leave-condition "angular counter mod $2\pi = 0$" is not sufficient. (iii) The agent cannot escape.

The following simpe Pledge-Algorithm (Algorithm 2.1) solves the problem. For the correctness proof we require structural properties.

Lemma 2.1 *The angular counter of the Pledge-Algorithm never attains a positive value.*

Proof.

- The angular counter is initialized by zero.
- After hitting an obstacle, the counter gets negative by the first right turn.
- The counter changes continuously, the agent leaves the obstacle if the counter gets zero again.

$\Rightarrow$ the statement is true.

□

Lemma 2.2 *If the agent does not leave the labyrinth by the Pledge-Algorithm, at the end there will be a finite path $\pi_\circ$, so that the agent follows this path infinitely often.*

Proof. The path generated by the Pledge-Algorithm is a polygonal chain. The set of vertices S stem from the vertex set of all obstacles plus vertices inside edges that occur when the algorithm leaves an obstacle at a vertex and hits an edges afterwards. The algorithm leaves an obstacle only at a vertex. Thus the overall number of possible vertices is bounded. If the Pledge leaves an obstacle at the same vertex twice, after that the same path $\pi_\circ$ will be used again and again, since the algorithm is deterministic and we left the same vertex with the same counter twice.

Therefore, if such path does not exists and the agent is not successful, finally the path will not leave an obstacle anymore. Obviously, the path $\pi_\circ$ exists in this case. □

Lemma 2.3 *Assume that the agent does not leave the labyrinth by the Pledge-Algorithm, the above mentioned finite path $\pi_\circ$ does not have self-intersections.*

Proof. An intersection can only occur at the boundary, since all movements (segments) in the *free space* are arranged in parallel. [1]

Let us assume that a part B of $\pi_\circ$ intersects with a former path A of $\pi_\circ$ at the boundary of some P_i. This means that there is a point $z \in P_i$ where B (running from the free space) hits A for the first time. The corresponding segment $e \in P_i$ points into a fixed direction. Therefore after turning, the turning angle, $C_B(z')$, of B at some point $z' \in e$ closely behind z and the turning angle, $C_A(z')$, differ by $2k\pi$ for some $k \in \mathbb{Z}$. If e cause turning angle $-\beta$ for B at z, we have $C_B(z') = -\beta$ and $C_A(z') = -\beta + 2k\pi$ with $k \in \mathbb{Z}$.

For $k > 0$ we have:

$$C_A(z') = -\beta + 2k\pi > -\pi + 2\pi = \pi \text{ ↯ by Lemma 2.1.}$$

For $k = 0$ it is clear that $C_A(z') = C_B(z')$ and A and B will never diverge, this contradicts the given situation that A and B are parts of $\pi_\circ$.

From $k < 0$ we conclude $C_A(p) < C_B(p)$ for all points p between z' and the first point z'', where the paths of A and B separate. From $C_A(p) < C_B(p)$ we conclude that at z'' the path B has $C_B(z'') = 0$ and leaves the obstacle. □

Finally, we conclude:

Theorem 2.4 *(Abelson, diSessa)*
For any given polygonal labyrinth, the Pledge-Algorithm will be able to escape from the labyrinth from any starting point, from which a successful path exists. *[Ad80]*

Proof. Assume, that the agent is not successful in the given situation. By Lemma 2.2 follows the finite path $\pi_\circ$ again and again, and the path $\pi_\circ$ has no self-intersections. Either the agent runs along $\pi_\circ$ in ccw-order or in cw-order. In ccw-order the angular counter will increase by $+2\pi$ in any round, which contradicts Lemma 2.1. In cw-order the angular counter will decrease by -2π in any round, thus at some point, the agent cannot leave an obstacle any more and in turn $\pi_\circ$ already belongs to a single obstacle. The agent follows the wall of an obstacle by Left-Hand-Rule and in cw-order, this can only mean that the agent is enclosed by the obstacle; see Figure 2.2(iii). There is no way out of the labyrinth from the given starting point. □

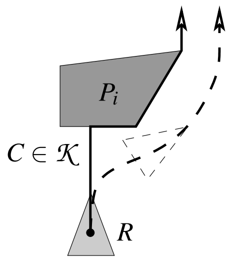
2.1.2 Pledge-Algorithm with sensor errors

The correctness proof of the Pledge-Algorithm (Theorem 2.4) make use of the assumptions that a point-shaped agent counts the angles without any errors and moves precisely into a specified starting direction. setzt voraus, dass der Roboter punktf"ormig ist und fehlerfrei arbeitet. As already mentioned physically the agent need not be a point, we can assume that the agent requires some room for its movement and in principle follows a curve calculated by the Pledge-Algorithm. The main problem is that the curve itself is not computed exactly by the robots abilities. For example the agent cannot precisely measure the turning angles at the boundary and cannot precisely follow a direction.

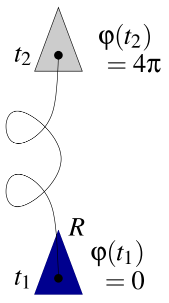
If the agent makes gross faults, we cannot assume that the pledge-like behaviour will succeed. Is there an error bound for the sensors that still allows to escape by a pledge-like behaviour?

[1] All segments that are not part of the boundary of some P_i.

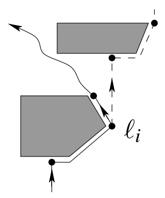
The general idea is as follows. We would like to design a class of curves $\mathcal{K}$ of the configuration space. Such a curve will be computed by an agent with sensor errors and imprecise measurements. Any curve from $\mathcal{K}$ represents a possible path for the escape. As mentioned above, the agent is guided by the computed curve and moves close to it. We would like to define sufficient conditions for the curves, such that the escape is guaranteed, if a corresponding path exist. The curve always have precise orientation. Its computation might be erroneous.



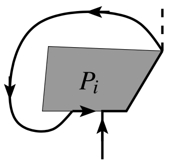
As the curves guides the agent, for convenience, we can consider curves in the configuration space by $C(t) = (P(t), \varphi(t))$ where $P(t) = (X(t), Y(t))$ is the location in the work space and $\varphi(t)$ the current turning angle. Note that two full turns will result in a turning angle of 4π instead of zero. Therefore $\varphi(t) \in \mathbf{R}$! A configuration (x, y, φ) is **half-free**, if the curve touches an obstacle in the work space and **free**, if no obstacle in the work space is met. The set of all half-free points is denoted by $\mathcal{C}_{\text{half}}$ whereas the set of all free points is denoted by $\mathcal{C}_{\text{free}}$. Points, where the curve moves from $\mathcal{C}_{\text{free}}$ to $\mathcal{C}_{\text{half}}$ are called Hit-Points, h_i. Points, where the curve leaves $\mathcal{C}_{\text{half}}$ and enters $\mathcal{C}_{\text{free}}$ are denoted as Leave-Points, ℓ_i. For simplicity, we will also denote the corresponding time parameter by h_i or ℓ_i, respectively.



The Pledge-Algorithm has two movement modi. Either the agent follows the wall and counts turning angles or the agent moves in the free space. Both movements can be erroneous, the agent diverges from the starting direction in the free space and drifts off or the agent cannot count turning angles precisely and will leave the obstacle earlier or later than in the original pledge path.



In principle we have to avoid that the agent moves in infinite loops. The figure shows that a large drift can easily result in a loop. The error of the agent is too large. But also a small deviation in the free space after each leave-point can sum up to a large total drift and an infinite loop. Figure 2.3 shows an example where there are small drifts after each leave that finally results in an infinite loop. This means that the direction in the free space should be globally stable.



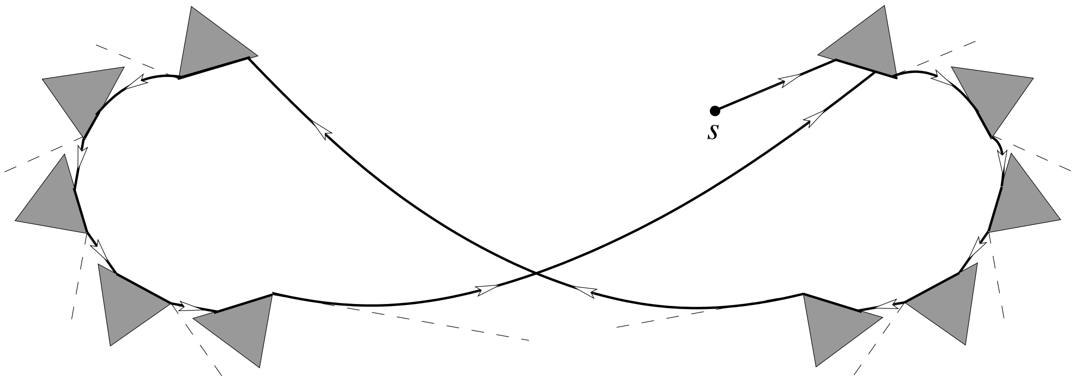


Figure 2.3: Small deviations sum up to a large deviation.

Additionally, for leaving the environment it is highly recommended that the agent at least moves into a certain global direction. One might also think that an erroneous compass will at least allow us to move generally into a half-plane. Therefore we require a $\mathcal{C}_{\text{free}}$-condition:

$$\forall t_1, t_2 \in C : P(t_1), P(t_2) \in \mathcal{C}_{\text{free}} \Rightarrow |\varphi(t_1) - \varphi(t_2)| < \pi$$

which guarantees that the angular counter in the free space maximally differs from the starting direction $\varphi(s)$ only by a fixed value. If we make use of a compass it seems to be reasonable to think that we can guarantee $\forall t : \varphi(t) \in]-\frac{\pi}{2}, +\frac{\pi}{2}[$ in the free space for starting direction zero.

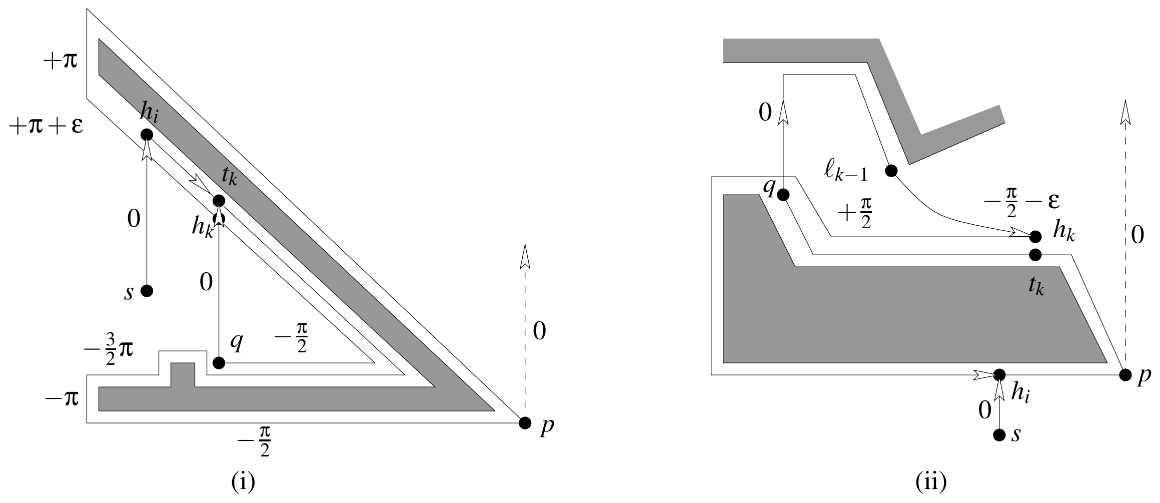


Figure 2.4: A local overturn of the angular counter can result in infinite loops.

Unfortunately, the $\mathcal{C}_{\text{free}}$-condition is not sufficient. We have to combine it with the angular counter at the obstacles. Figure 2.4 shows two examples where the agent overturns the counter at the obstacle for a while (because of sensor errors) but obtains overall precise values later for the leaving condition. This has nothing to do with the free space condition. The infinite loop happens at the obstacle. In this case the agent passes the first true leave-point p and leaves the obstacle later at a point q which is also a legal leave-point.

In Figure 2.4(i) by the overturn the curve mets the same obstacle h_k at again. In Figure 2.4(ii) by the overturn the agent first visit another obstacle but then returns via some leave-point ℓ_{k-1} again to the starting obstacle at h_k. In both case $P(h_k)$ is visited twice and during the first visit at t_k the angular counter was overturned. In the left hand side figure the angular counter at t_k is larger than π, in the right hand side figure the counter at t_k is $+\frac{\pi}{2}$. Figure 2.4(ii) shows a second error source. Since the angular counter at h_k is $-\frac{\pi}{2}-\varepsilon$ both errors also sum up to an error larger than π. We state that for the hit-point h_k the angular counter of a previous visit was overturned. Together with the error at h_k there is an overall overturn larger than π.

We subsume the requirement in the $\mathcal{C}_{\text{half}}$-condition:

$$\forall h_i, t \in C : P(t) = P(h_i) \Rightarrow \varphi(t) - \varphi(h_i) < \pi .$$

We can also think about the compass with a deviation of $\pi/2$ into both directions. This means that we can overturn the angle counter at the obstacle by less than $\pi/2$. Together with a deviation of less than $-\pi/2$ in the free space we might hit this point again but $\varphi(t) - \varphi(h_i) < \pi$ holds.

Definition 2.5 Let $\mathcal{K}$ be a class of curves in $\mathcal{C}_{\text{free}} \cup \mathcal{C}_{\text{half}}$ such that any curve from $\mathcal{K}$ fulfils the following conditions:

(i) Parameterized pledge like curve with turn-angles and position:
$C(t) = (P(t), \varphi(t))$ with $P(t) = (X(t), Y(t))$
(ii) At the boundary the curve surrounds obstacles by Left-Hand-Rule.
(iii) Any leave-point is a vertex of an obstacle.
(iv) $\mathcal{C}_{\text{free}}$-condition holds: $\forall t_1, t_2 \in C : P(t_1), P(t_2) \in \mathcal{C}_{\text{free}} \Rightarrow |\varphi(t_1) - \varphi(t_2)| < \pi$
(v) $\mathcal{C}_{\text{half}}$-condition holds: $\forall h_i, t \in C : P(t) = P(h_i) \Rightarrow \varphi(t) - \varphi(h_i) < \pi$

Obviously, the path constructed by the error-free Pledge-Algorithm does belong to $\mathcal{K}$. For the correctness proof that any curve of $\mathcal{K}$ that is constructed in a pledge-like fashion will escape from the labyrinth. We show some important structural properties of curves in $\mathcal{K}$.

Lemma 2.6 *A curve $C \in \mathcal{K}$ has no self-intersection.*

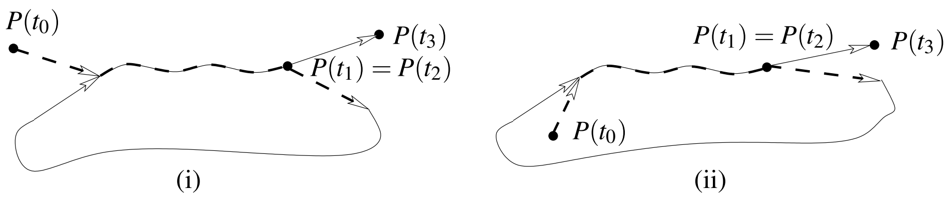


Figure 2.5: The difference between (i) crossing and (ii) touching at t_2.

Note that a curve from $\mathcal{K}$ might touch itself; see Figure 2.5.

Proof. Assume that C has an intersection, consider the first intersection. There are parameter t_1 and t_2 with $t_1 < t_2$ and $P(t_1) = P(t_2)$ and the first intersection occurs at $P(t_2)$.

This means that between t_1 and t_2 there is a cw or ccw turn. If the intersection lies in the free space, obviously the $\mathcal{C}_{\text{free}}$-condition does not hold. Thus we can assume that $P(t_2)$ is in $\mathcal{C}_{\text{half}}$ holds. Consider the case of a cw loop as depicted in Figure 2.6.

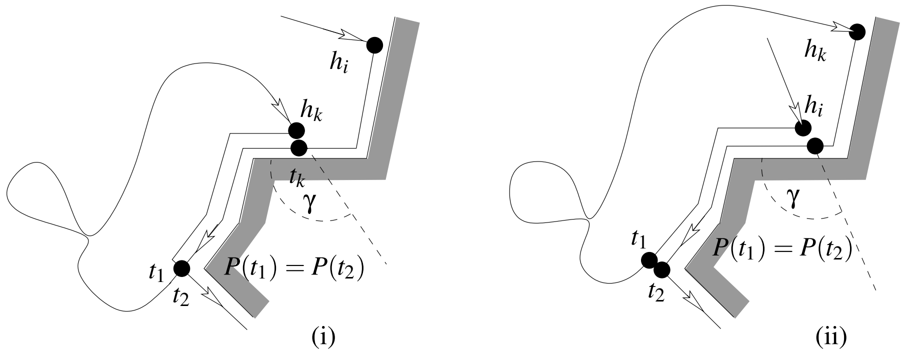


Figure 2.6: Cw loop and two cases.

The curve hits the obstacle at h_i, follows the wall, leaves the obstacle at time t_2 comes back to the obstacles at $h_k > h_i$ and has an intersection at t_2; see Figure 2.6(i). If the point $P(h_k)$ was not visited between h_i and t_1 there is only a touching event at t_2; see Figure 2.6(ii).

Let $\varphi(h_k^+)$ deonte the angular counter after the agent turns into the direction of the corresponding edge. we have $\varphi(h_k^+) = \varphi(h_k) + \gamma$, where γ denotes the turning angle at the edge. We have $-\pi < \gamma < 0$. Additionally, by the full cw turn we conclude $\varphi(h_k^+) = \varphi(t_k) - 2\pi$. Also the $\mathcal{C}_{\text{half}}$-holds and we obtain:

$$\begin{aligned} & \varphi(t_k) - \varphi(h_k) < \pi \\ \Leftrightarrow\ & \varphi(h_k^+) + 2\pi - \varphi(h_k) = \varphi(h_k) + \gamma + 2\pi - \varphi(h_k) < \pi \\ \Leftrightarrow\ & \gamma < -\pi \ \lightning \end{aligned}$$

Similar arguments hold for a loop in ccw order, which is an exercise. The first intersection cannot exists. By induction there is no intersection at all. □

Lemma 2.7 *A curve $C \in \mathcal{K}$ hits any edge of the environments at most once.*

Proof. Assume that a single edge e has two hit-points. After the first hit h_i of edge e the curve can only leave the obstacle at a vertex and then comes back to e at h_k; see Figure 2.7.

In $P(h_i)$ and $P(h_k)$ the agent turns clockwise in order to follow the edge e which gives turning angles $-\pi < \gamma_i, \gamma_k < 0$. Let $\varphi(h_i^+)$ and $\varphi(h_k^+)$ denote the turning angles as in the proof of Lemma 2.6 (turning angle directly after the hit-point). Let w. l. o. g. $\varphi(h_i^+) = 0$.

In h_i^+ and h_k^+ the curve follow the edge e, and the direction differs only by $2j\pi$ for some $j \in \mathbb{Z}$ This means $\varphi(h_k^+) = 2j\pi, j \in \mathbb{Z}$.

For $j \neq 0$, and with $\varphi(h_i) = -\gamma_i$ and $\varphi(h_k) = \varphi(h_k^+) - \gamma_k$ we conclude $|\varphi(h_k) - \varphi(h_i)| = |2k\pi - \gamma_k + \gamma_i| > \pi$, which contradicts to the $\mathcal{C}_{\text{free}}$-condition.

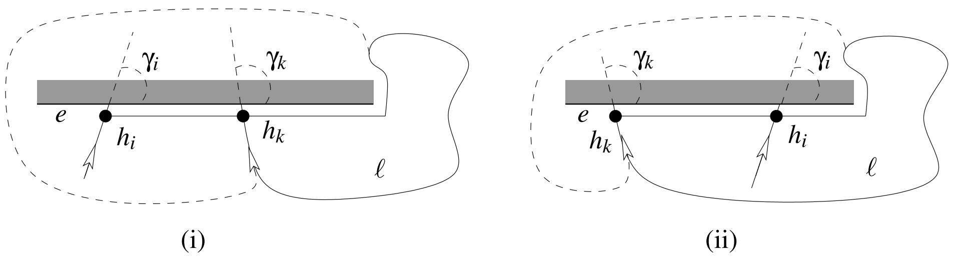


Figure 2.7: A curve from $\mathcal{K}$ hits any edge once.

Therefore we conclude $j = 0$ and also $\varphi(h_k^+) = 0$. But we can argue that there be a full cw or ccw turn from $P(h_i)$ to $P(h_k)$ without intersections; see Figure 2.7. The curve C made a full turn with angular counter change of $\pm 2\pi$. This means that $\varphi(h_k^+) = \pm 2\pi$ should hold. ↯ □

By Lemma 2.7 we can now prove that the condition from Definition 2.5 are sufficient. First, we require a helping lemma. If the curve gets stuck onto a single obstacle, the obstacle should enclose the curve.

Lemma 2.8 *If a curve $C \in \mathcal{K}$ does not leave an obstacle anymore the curve is enclosed by the obstacle.*

Proof. If the curve does not leave an obstacle after the last hit-point, the path along the boundary is repeated infinitely often. The path can be in cw or in ccw order which means a counter change of $+2\pi$ for ccw order or a counter change of -2π in cw order for any round. In the first case at some point the counter gets arbitrarily large for any point on the boundary. So also for the last hit-point and the $\mathcal{C}_{\text{half}}$-condition is violated. This means we can only have a cw order loop and by the Left-Hand-Rule the curve has to be enclosed by the obstacle. □

Theorem 2.9 *(Kamphans, Langetepe, 2003)*
An agent, who follows a path from $C \in \mathcal{K}$ will escape from any labyrinth and from any position, if an escape path exists. *[KL03]*

Proof. If there is an escape path the agent and the curve is not enclosed by an obstacle. Therefore the curve $C \in \mathcal{K}$ will leave any obstacle after a while. Since any edge is hit at most once by Lemma 2.7 there will be no hit any more after a while. The $\mathcal{C}_{\text{free}}$-condition takes care that the agent steadily moves into a halfplane w.r.t. a given direction. Thus we will escape from the environment. □

2.1.3 Applications

We would like to consider the impact of Theorem 2.9. Let us first assume that we make use of a compass for counting the total turns around the obstacles and for holding the direction in $\mathcal{C}_{\text{free}}$. If the deviation from the starting direction is never larger than $(-\pi/2, \pi/2)$, such a compass will help us leaving the labyrinth.

In this case in $\mathcal{C}_{\text{free}}$ we can guarantee an angular range $(-\frac{\pi}{2}, +\frac{\pi}{2})$. Along the boundary the absolut error is smaller than $|\frac{\pi}{2}|$, the maximal positive value along the boundary is smaller than $+\frac{\pi}{2}$, the minimal value in $\mathcal{C}_{\text{free}}$ is larger than $+\frac{\pi}{2}$, this yields the $\mathcal{C}_{\text{half}}$-condition.

$$\forall h_i, t \in C : P(t) = P(h_i) \Rightarrow \varphi(t) - \varphi(h_i) < \pi\,!$$

Corollary 2.10 *The usage of a compass with absolut deviation smaller than $\frac{\pi}{2}$ will help to leave a labyrinth by a pledge like algorithm.*

Next assume that we would like to make use of small deviations of the environment itself. Therefore, we consider obstacles that consists of axis-parallel edges, only. For such orthogonal polygons, we can simply sum up the reflex vertices (inner angle $> \pi$, $+1$) and the convex vertices (inner angle $< \pi$, -1) as indicated and will leave an obstacle with counter value 0; see Figure 2.8(i). If we guarantee a deviation in the range $(-\frac{\pi}{2}, +\frac{\pi}{2})$ in the free space, we can successfully apply the pledge algorithm. After a hit, we only have to find out whether the edge is horizontal or vertical. For vertical edges we simply slip along the edge and wait for the next hit. Thus we start the movement along the boundary with angular counter $+1$, following a horizontal edge.

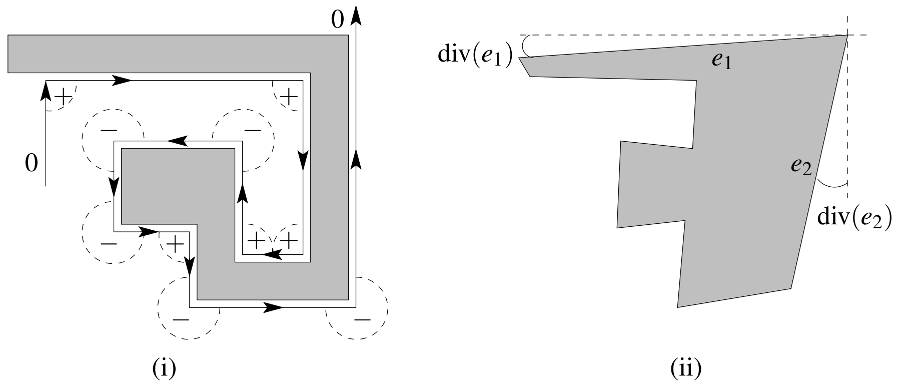


Figure 2.8: (i) Counting the angles in orthogonal polygons, (ii) pseudo-orthogonal polygon with deviation div.

More generally for such a counting argument we have to take care that: Folgende drei Bedingungen sind also einzuhalten:

1. Reflex and convex vertices can be distinguished: Count the rotation correctly.
2. Maximal deviation from the starting direction. Interval of size π.
3. Distinguish: Horizontal/verticale edge.

Now let us assume that the polygons are not exactly axis-parallel but roughly as shown in Figure 2.8(ii).

By, $\text{div}(e)$, for edge Kante $e = (v, w)$ we define the smallest deviation from a vertical or horizontal edge passing through w or v. This deviation should be small in total for all edges. Additionally, we would like to take care that the number of reflex and convex vertices of the polygon fits to an axis-parallel polygon.

Definition 2.11 A polygonal scene is δ-pseudo orthogonal for $\delta > 0$, if for any polygon we have (number if convex vertices) = (Number of reflex vertices) + 4 and $\text{div}(P) := \max_{e \in P} \text{div}(e) \leq \delta$ holds.

For δ-pseudo orthogonal we would like to proceed as indicated above and have to fulfil the above three conditions. Let us additionally assume that the angle counter device on the obstacle has a deviation (error) of no more than ρ. The following corollary shows some legal value for ρ and δ also with the interplay of the free space movements.

Corollary 2.12 *Let P be a δ-pseudo orthogonal scene and let us assume that we count the angles with precision ρ such that $\delta + \rho < \frac{\pi}{4}$. If we take care that the deviation from the starting direction in $\mathcal{C}_{\text{free}}$ is no more than $\frac{\pi}{4} - 2\delta - \rho$, the simple reflex/convex counter pledge like algorithm helps us to leave a labyrinth.*

Proof. We have to distinguish between reflex and convex vertices at the boundary, otherwise the $+1$, -1 counting will be erroneous. We consider the worst-case situation for our measurement. Let us assume that at a vertex we measure the outer angle γ as shown in Figure **??**. For $\gamma < \pi$ we assume that we have a reflex vertex and for $\gamma > \pi$ we assume that the vertex is convex.

We would like to guarantee a correct detection for tha maximal error and deviation. The correct angle of a convex vertex is $\frac{3}{2}\pi$ and the correct angle of a reflex vertex is $\frac{\pi}{2}$. Therefore we require:

$$\begin{array}{lcl} \frac{3}{2}\pi - 2\delta - \rho > \pi & \text{und} & \frac{\pi}{2} + 2\delta + \rho < \pi \\ \Leftrightarrow 2\delta + \rho < \frac{\pi}{2} & & \Leftrightarrow 2\delta + \rho < \frac{\pi}{2}\,. \end{array}$$

Additionally, we would like to distinguish between horizontal and vertical edges after hitting an edge. Either we slip along the vertical edge or we start at the horizontal with the simple counter. Again we assume the worst-case situation; see Figure 2.9.

We measure the turning angle γ for the corresponding edge. For a horizontal edges this is exact $-\frac{\pi}{2}$; see Figure 2.9(i). If this angle is between $-\frac{\pi}{4}$ and $-\frac{3\pi}{4}$ we conclude that we have a horizontal edge. Otherwise the edge is assumed to be vertical. Note that γ is always negative. We assume that the deviation from the starting direction is φ.

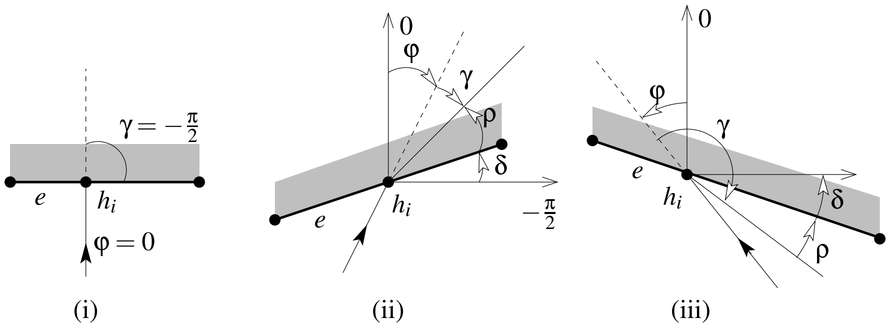


Figure 2.9: Hitting a horizonal edges (i) in the error-free case, (ii) for small absolut γ, (iii) for large absolut γ.

In Figure 2.9(ii) the deviations φ, δ and ρ should make $|\gamma|$ as small as possible and still smaller than $-\frac{\pi}{4}$, φ is negative. In Figure 2.9(iii) the deviations φ, δ and ρ should make $|\gamma|$ as large as possible and larger than $-\frac{3\pi}{4}$, φ is positive. We conclude from Figure 2.9(ii)

$$\gamma = -\frac{\pi}{2} - \varphi + \delta + \rho < -\frac{\pi}{4} \;\Leftrightarrow\; -\frac{\pi}{4} + \delta + \rho < \varphi\,,$$

and from Figure 2.9(iii)

$$\gamma = -\left(\frac{\pi}{2} + \varphi + \delta + \rho\right) > -\frac{3}{4}\pi \;\Leftrightarrow\; \frac{\pi}{4} - \delta - \rho > \varphi\,.$$

We We detect horizontal edges precisely if $\varphi(h_i) \in \left]-\frac{\pi}{4} + \delta + \rho, \frac{\pi}{4} - \delta - \rho\right[$ holds. Therefore we require $\delta + \rho < \frac{\pi}{4}$. A maximal deviation of $\frac{\pi}{4} - \delta - \rho$ would be enough for correct detections. Since we might start the free space move with an error of δ at a vertex we require $\frac{\pi}{4} - 2\delta - \rho$ for the deviation. □

Exercise 15 *In the above corollary we can set $\delta = 0$ and $\rho = 0$ and require that we do not deviate in the free space by an angle of $\pi/4$. Why is this different to the error-free case where an error of less than $\pi/2$ was allowed for the free space movements.*

2.2 Navigation with touching sensor

We distinguish between the term Navigation for visiting a given target (known coordinates) and the term Searching for searching for an unknown goal (unknown coordinates). The family of the so-called Bug-Algorithms are the first algorithms for the navigation task in polygonal environments[2]. The first

[2]In this case Bug is not meant as a synonym for an error.

simple strategies have been introduced by Lumelsky and Stepanov [LS87], extensions and modifications came from Sankaranarayanan et al. [SM92, SV90a, SV90b, SV91]. Many variants have been discussed since then. Bug-variants have been practically used for the navigation of some of the Mars rovers like Sojourner or Bridget, (see also RoverBug, [LB99]).

In the following we assume that the coordinates of the target are known and that the agent have a finite storage so that coordinates of points and /or length of (sub)path can be stored. The agent also is aware of the coordinates of its current position, for example by GPS.

Any Bug-algorithm runs with the same principle and actions: The agent moves toward the target until an obstacle is visited (Move-To-Target action) Then the agent follows the wall of the obstacle for a while (Follow-Wall action) until some condition triggers the next movement in the free space toward the target. The leaving condition is the main difference between the Bug-variants.

We assume that the agent R is point-shaped and equipped with a touch sensor for the Follow-Wall action. We make use of the following notations:

- $|pq|$ denotes the distance between two points p and q,
- $D := |st|$ denotes distance from start s to target t,
- π_S denotes the path of a strategy S from s to t; $|\pi_S|$ denotes the length of this path where $|\pi_S| := \infty$, if there is no such path,
- $\mathrm{U}(P_i)$ denotes the perimeter of the obstacle P_i.

2.2.1 Strategies of Lumelsky and Stepanov

The first algorithm Algorithm 2.2, Bug1, leaves an obstacle P_j at a point $p \in P_i$ that is the closest point to the target. This defines a sequence of **Hit-Points** h_i, where the agent hits an obstacle and **Leave-Points** ℓ_i, where the agent leaves an obstacle. Since the coordinates of the target and the coordinates of the current position are known, the agent can calculate the corresponding distances. Additionally, by successively counting small steps, the agent can calculate the path length of the path along the boundary during the circumnavigation and also the path length to the currently computed optional leave-point. Additionally, the path length (along the boundary) to the With these values the agent can perform step 3 of Algorithm 2.2 Figure 2.10 shows an example for the path of Bug1.

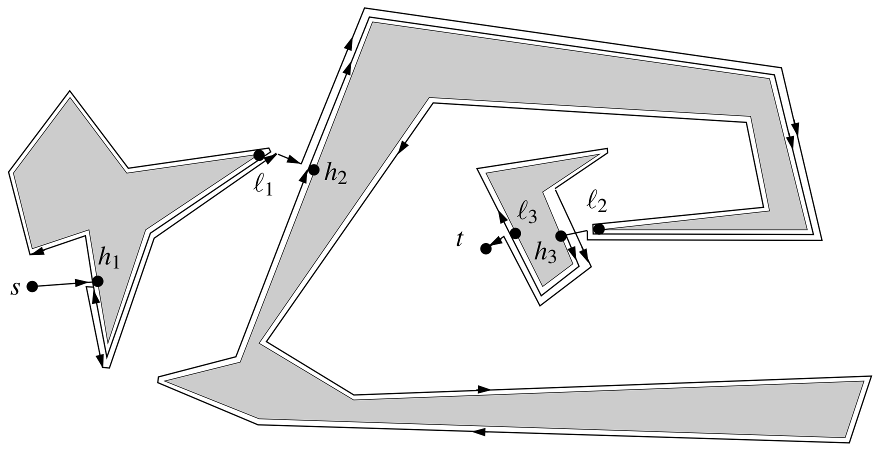


Figure 2.10: Example execution of strategy Bug1.

We assume that there is a finite number of polygonal obstacles and that the obstacles do not touch or intesect. The number of polygons is finite in the sense that any circle of fixed radius contains only finitely many *obstacles P_i*.

Algorithm 2.2 Bug1

```
0. $\ell_0 := s$, $i := 1$
1. From $\ell_{i-1}$ move toward the target, until
   (a) Target is visited: Stop!
   (b) An obstacle is reached at point $h_i$. If $h_i = \ell_{i-1}$: The goal cannot be reached.
2. Surround the obstacle in cw-order — keep track of the point $\ell_i$ on the boundary with the shortest
   distance to $t$ —, until
   (a) Target is visited: Stop!
   (b) $h_i$ is reached.
3. Move along the shortest path along the boundary to $\ell_i$.
4. Increase $i$, GOTO 1.
```

Theorem 2.13 *(Lumelsky, Stepanov, 1985)*
Strategy Bug1 finds a path from a starting point s to a target t, if such a path exists. *[LS87]*

Proof. For the sequence of hit- und leave-points we have

$$|st| \geq |h_1t| \geq |\ell_1t| \ldots \geq |h_kt| \geq |\ell_kt|.$$

Since for any visited obstacles we choose a leave-point that is closest to the target, any obstacles can be left. Otherwise, if this is not the case, the obstacle would fully enclose the target. This also means that we have a strict $>$ in the above sequence. Any obstacle can be visited only once and the finite number of obstacles within the circle of radius $|st|$ around t lead to a finite sequence which ends at the target. □

For the performance we conclude:

Theorem 2.14 *(Lumelsky, Stepanov, 1985)*
Let π_{Bug1} denote the path from s to t, for the successful application of the strategy Bug1. *[LS87] We have:*

$$|\pi_{\text{Bug1}}| \leq D + \frac{3}{2}\sum_i \mathrm{U}(P_i).$$

Proof. We subdivide the path into the movements along the boundary of the obstacles P_i and the movements in the free space. Since step 3. of Algorithm 2.2 makes use of a shortest path we have path length $\frac{3}{2}\sum \mathrm{U}(P_i)$ for any visited obstacle. It remains to calculate the length D' for the free space movements. We show that $D' \leq D$ holds.

$$\begin{aligned}
D' &= |sh_1| + |\ell_1h_2| + \ldots + |\ell_{k-1}h_k| + |\ell_kt| \\
&\leq |sh_1| + |\ell_1h_2| + \ldots + |\ell_{k-1}h_k| + |h_kt| \\
&= |sh_1| + |\ell_1h_2| + \ldots + |\ell_{k-1}t| \\
&\ldots \\
&\leq |sh_1| + |\ell_1h_2| + |h_2t| \\
&= |sh_1| + |\ell_1t| \\
&\leq |sh_1| + |h_1t| = |st| = D
\end{aligned}$$

□

We can compare the above result with the lower bound Theorem **??** and conclude that in comparison to any other Bug-strategy the strategy Bug1 can be consodered to be $\frac{3}{2}$-competitive.

Corollary 2.15 *Bug1 is $\frac{3}{2}$-competitive in comparison to arbitrary Bug-like online strategyies.*

In the next variant we would like to avoid complete circumnavigations of the obstacles. Therefore we make use of a line G passing through the segment st. At any time during the Wall-Follow action we will try to move toward the target if we reach a point at G that is closer to t than the previous hit-point; see Algorithm 2.3. Note that by this action, it is possible to visit an obstacles more than once which was impossible for Bug1. h_j and ℓ_j do no longer denote hit- and leave-points of the j-th obstacle.

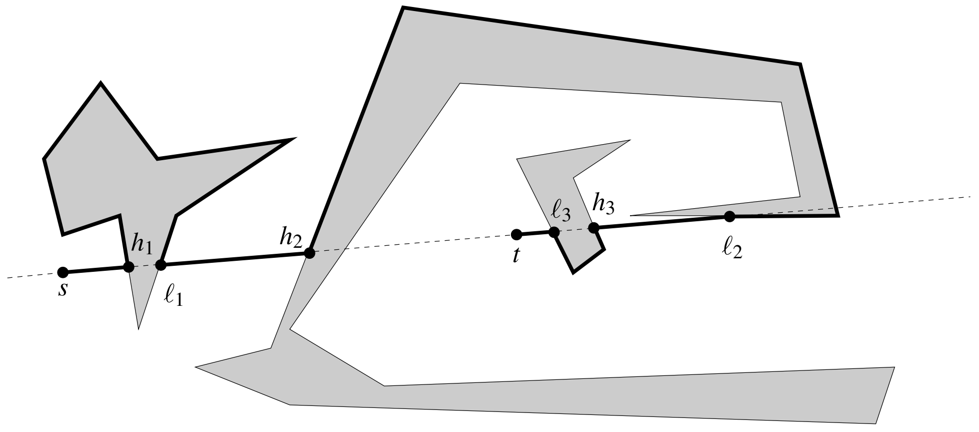


Figure 2.11: Example of the execution of the strategy Bug2.

Figure 2.12 shows an example, where an obtacle is visited more than once. After hit-point h_3 the agent does not leave the obstacle at p_1 or ℓ_1 since $|h_3t|$ is smaller thanthe distance of p_1 and ℓ_1 to t. At p_2 and p_3 the agent cannot leave the obstacle since the segments $\overline{p_{2/3}t}$ are blocked by the obstacle.

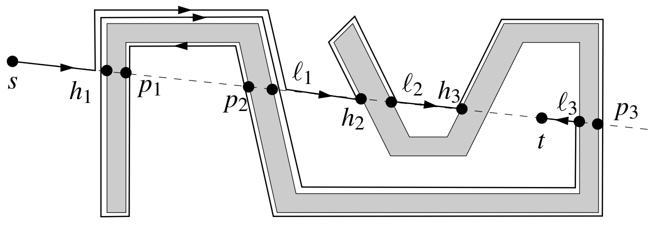


Figure 2.12: The execution of Bug2 can lead to several visits of the same obstacle.

The number of polygons is finite in the sense that any circle of fixed radius contains only finitely many *obstacles P_i.*

Lemma 2.16 *The strategy Bug2 meets finitely many obstacles.*

Proof. In step 2b of Algorithm 2.3 the agent leaves an obstacle only if $|\ell_jt| < |h_jt|$ holds. Since the circle of radius $|\overline{st}|$ around t contains only finitely many obstacles we can hit only finitely many obstacles. □

The number of surroundings depend on the intersections of the line passing through st with the boundary of the relevant obstacles.

Lemma 2.17 *Let n_i denote the number of intersections of the line $\overleftrightarrow{st}$ passing through st with the boundary of polygon P_i. The strategy Bug2 visits any point of P_i only $\frac{n_i}{2}$ times.*

Proof.

Bug2 successively defines pairs (h_j, ℓ_j) of hit- und leave-points and by the leave condition we have

$$|h_jt| > |\ell_jt| > |h_{j+1}t|.$$

Algorithm 2.3 Bug2

```
0. $\ell_0 := s$, $j := 1$
1. From $\ell_{j-1}$ move toward the target, until
    (a) Target is reached: Stop!
    (b) An obstacle is visited at $h_j$.
2. Surround the obstacle in cw-order, until
    (a) Target is reached: Stop!
    (b) The line passing segment $st$ is visited at point $q$, $|qt| < |h_j t|$ and $\overline{qt}$ is free, such that we can leave the obstacle from $q$ toward the target.
        Set $\ell_j := q$, $j := j+1$ and GOTO 1.
    (c) $h_j$ is visited again without reaching a point $q$ as in described in b). The target cannot be reached. erreicht werden.
```

This means that any point is only once a leave-point or a hit-point and any intersection point can appear only in one pair (h_j, ℓ_j). On the other hand a single pair can only lead to one full surrounding, if the same hit-point is visited, the strategy stops. We have at most $\frac{n_i}{2}$ pairs and surroundings. □

Finally we conclude that we have only finitely many relevant intersections and either the strategy visits a current hit-point again and the corresponding obstacle enlcoses the target or we will finally succeed.

Corollary 2.18 *Strategy Bug2 is successful, if the target can be reached.*

The performance of Bug2 is given in the following statement.

Theorem 2.19 *(Lumelsky, Stepanov, 1985)*
Let π_{Bug2} denote the path from s to t, for the successful application of strategy Bug2. We have

$$|\pi_{\text{Bug2}}| \leq D + \sum_i \frac{n_i \, \mathrm{U}(P_i)}{2}.$$

Here P_i is an obstacle that is visited during the execution of Bug2. [LS87]

Proof. The term $\sum \frac{n_i \mathrm{U}(P_i)}{2}$ follows from Lemma 2.17. For the length of the free space movements, say D', between the obstacles, we make use of the same arguments as in the proof of Theorem 2.14 and conclude $D' \leq D$. □

Bug2 is not always better than Bug1. Obviuously, in the presence of convex obstacles, Bug2 outperforms Bug1.

Corollary 2.20 *For a polygonal scene with convex obstacles the successful application of strategy Bug2 has path length*

$$|\pi_{\text{Bug2}}| \leq D + \sum_i \mathrm{U}(P_i).$$

Exercise 16 *Compare the variants Bug1 and Bug2. Present an example where Bug1 outperforms Bug2. Show that for both strategies the performance guarantee is tight.*

2.2.2 Strategies from Sankaranarayanan and Vidyasagar

Many variants of the Bug-strategies have been discussed. Many of them make use of more sensor power for local improvement. For example a VisBug2 strategy makes use of a visibility range and can find local short-cuts for the Bug2 path. We would like to mention some structural different variants from Sankaranarayanan and Vidyasagar. The reason is that we would like to show that some local optimization can have unexpected disadvatages.

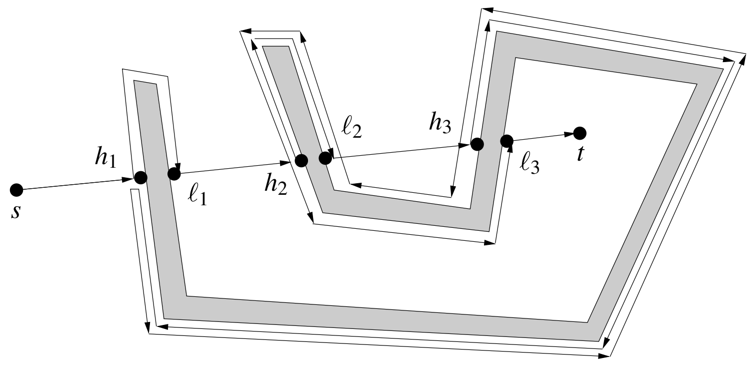


Figure 2.13: Example of the execution of Change1.

Bug1 fully surrounds any obstacle, Bug2 tries to avoid this by moving toward the goal a bit earlier. In this case a single obstacle can be visited many times. Algorithm 2.4 tries to avoid this behaviour: If a surrounding is started, and an old hit- or leave-point (not the current hit-point!) is visited, the strategy starts moving along the boundary in ccw-order; see Figure 2.13.

Theorem 2.21 *(Sankaranarayanan, Vidyasagar, 1990)*
For the length of the path of the successful application of strategy Chang1 we have *[SV90a]*

$$|\pi_{\text{Change1}}| \leq D + 2 \cdot \sum_i \mathrm{U}(P_i).$$

Proof. Exercise □

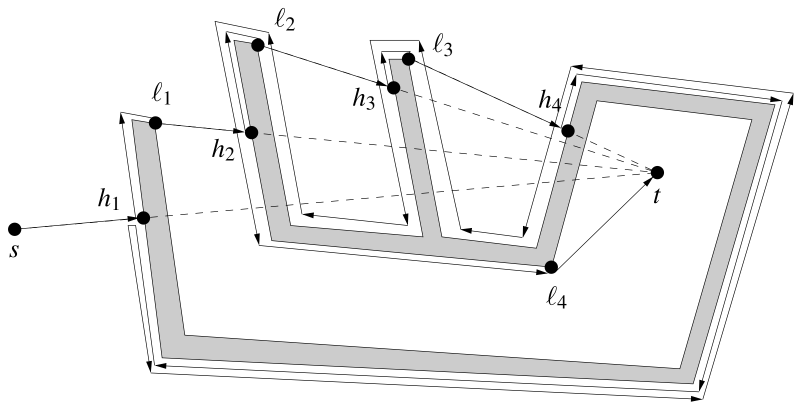


Figure 2.14: Example execution of strategy Change2.

Strategy Change2 (Algorithm 2.5) differs from Change1 only in the leaving condition. The leave-point is not restricted to a point on the line $\overleftrightarrow{st}$. As soon as there is a point q on the boundary in the Follow-Wall action that is closer to the target than the distance $|ht|$ for the last hit-point,we will leave the obstacle toward the target, if this is possible. Note that such a behaviour can also be used for a variant of Bug2.

Theorem 2.22 *(Sankarayanan, Vidyasagar, 1990)*
For the length of the path of the successful application of strategy Chang1 we have *[SV90b]*

$$|\pi_{\text{Change2}}| \leq D + 2 \cdot \sum_i \mathrm{U}(P_i).$$

Proof. Exercise □

Exercise 17 *Present proofs for the above two Theorems. Show that the bounds are tight.*

Algorithm 2.4 Wechsel1

```
0. $\ell_0 := s$, $i := 1$
1. Move from $\ell_{i-1}$ along the line passing $s$ and $t$ toward the target, until
    (a) Target is reached: Stop!
    (b) An obstacle is reached at $h_i$.
2. Surround the obstacle, until
    (a) Target is reached: Stop!
    (b) The line passing $s$ and $t$ is visited a some point $q$ such that the distance from $q$ to $t$ is smaller
        than $h_it$ and the segment $\overline{qt}$ is *free* (see refalgobug2).
        Set $\ell_i := q$, $i := i+1$ und GOTO 1.
    (c) A hit- or leave-point $h_j$ or$\ell_j$ with $j < i$ is visited: Move back to $h_i$ in ccw-order and start
        ccw-order surrounding under condition (a), (b) oder (d) (not (c) again!)
    (d) $h_i$ is visited again without reaching a point as indicated in (b) or (c). The goal is enclosed by
        an obstacle.
```

Algorithm 2.5 Wechsel2

As Change1, but:

```
0. $\ell_0 := s$, $i := 1$
1. Move from $\ell_{i-1}$ along the line passing $s$ and $t$ toward the target, until
2.(b) A point $q$ is visited such that the distance from $q$ to $t$ is smaller than $h_it$ and the segment $\overline{qt}$ is *free*
      (see refalgobug2).
      Set $\ell_i := q$, $i := i+1$ und GOTO 1.
```

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